

Numeric Simulation of wave propagation in diverse media using finite-difference approach

Simulación numérica de la propagación de ondas en diversos medios usando diferencias finitas

PhD. Carlos Arturo Parra Ortega ¹, PhD. José Orlando Maldonado Bautista ¹
MSc. Luis Armando Portilla Granados ¹

¹ Universidad de Pamplona, Facultad de Ingenierías y Arquitectura, Ingeniería de Sistemas - Grupo de Investigación CICOM, Pamplona, Norte de Santander, Colombia.

Correspondence: carapa@unipamplona.edu.co

Received: October 15, 2023. Accepted: December 17, 2023. Published: March 16, 2024.

How to cite: C. A. Parra Ortega, J. O. Maldonado Bautista, and L. A. Portilla Granados, "Numeric Simulation of wave propagation in diverse media using finite-difference approach", RCTA, vol. 1, no. 43, pp. 86–91, Mar. 2024.
Recovered from <https://ojs.unipamplona.edu.co/index.php/rcta/article/view/2823>

Copyright 2024 Colombian Journal of Advanced Technologies.
This work is licensed under a [Creative Commons Attribution-NonCommercial 4.0 International License](https://creativecommons.org/licenses/by-nc/4.0/).



Abstract: The propagation of mechanical waves is a natural or artificial phenomenon and is transmitted in a medium; its propagation is modeled using partial differential equations, which must be solved numerically. The derivatives with respect to time and space are solved using a second-order approximation through centered finite difference operators. Because the modeling is carried out in depth, the values of the spatial axes are considered positive when depth increases. Also considering velocities and efforts within a stepped mesh, which takes into account the deformation generated in the medium due to the efforts and the impact of the wave in such medium. This deformation effect will be analyzed mathematically taking into account the Lamé coefficients, given that the medium through which the wave propagates is isotropic. Reflection and transmission of the wave will also be analyzed to look at its natural behavior. Since the wave modeling is computational, a treatment is made to the conditions of stability and numerical dispersion to avoid obtaining erroneous results and to be able to visualize the wave with a more realistic behavior. Edge absorption methods were analyzed to avoid the visualization of false reflections not existing in the elastic medium.

Keywords: Finite differences, elastic medium, deformation, seismic modeling.

Resumen: La propagación de las ondas mecánicas es un fenómeno natural o artificial y se transmite en un medio; su propagación se modela utilizando ecuaciones diferenciales en derivadas parciales, las cuales deben resolverse numéricamente. Las derivadas respecto al tiempo y al espacio se resuelven utilizando una aproximación de segundo orden a través de operadores en diferencias finitas centrados. Debido a que el modelado se realiza en profundidad, los valores de los ejes espaciales se consideran positivos cuando esta aumenta. Considerando además velocidades y esfuerzos dentro de una malla escalonada, la cual tiene en cuenta la deformación generada en el medio debido a los esfuerzos y el impacto de la onda en dicho medio. Este efecto de deformación se analizará matemáticamente teniendo

en cuenta los coeficientes de Lamé, dado que el medio por donde se propaga la onda es isotrópico. También se analizará reflexión y transmisión de la onda para mirar su comportamiento natural. Puesto que el hecho de que el modelado de la onda es computacional, se hace un tratamiento a las condiciones de estabilidad y dispersión numérica para no obtener resultados erróneos y ser capaces de visualizar la onda con un comportamiento más realista. Los métodos de absorción de borde fueron analizados para evitar la visualización de falsas reflexiones no existentes en el medio elástico.

Palabras clave: Diferencias Finitas, medio elástico, tensión, deformación, modelamiento de fenómenos sísmicos.

1. INTRODUCTION

The finite difference (FD) method is traditionally used to solve the equation of acoustic and elastic waves, these equations describe the movement of the wave by using partial derivatives of both the spatial and temporal components. The methods allow the simulation of wave propagation in highly complex heterogeneous media and geologically elastic media. The most used FD operators are second-order centered differences. This method shows that it satisfies the conditions of stability and numerical dispersion.

This document is organized as follows: the mathematical formulation is shown in section 2, and the computational implementation in section 3. Section 4 shows the experimentation and section 5 the results of the implementations. Finally, conclusions and references are presented.

2. MATHEMATIC FUNDAMENTALS

In order to model the phenomenon of seismic wave propagation, it is necessary to first describe the forces and stresses that occur in the subsoil when is applied an energy pulse that causes deformation. The excitation source or energy pulse in an elastic medium does not generate an instantaneous deformation in the particles that constitute it, but such deformation requires an amount of time to propagate in different directions from its point of origin and traveling various lengths of the disturbed medium. This phenomenon is well known from physical experiences and events that are observed daily in our environment, such as waves on the water surface, waves on the ropes caused by earthquakes or explosions, vibration of a membrane, among others. These are some of the examples that allow us to understand the propagation of undulating mechanical movements.

The elements that constituting the medium experiment a deformation that allows the transmission of the perturbation (in this case, in energy that is transmitted from one particle to the other) from one point to another, in this way the wave advances through the medium, as can be seen in figure 1.

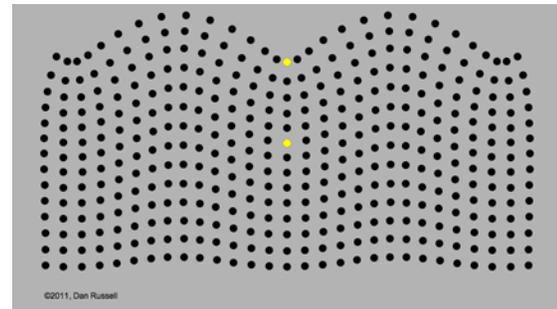


Fig. 1. Deformation of particles during the transmission of a transversal wave (Cerjan [1])

In this process, the perturbation has to overcome the resistance of the medium to being deformed, as well as the resistance to movement due to inertia. This transmission of energy is carried out by the transmission of motion from one particle to the other and not by the transmission of the medium as a whole.

Its expression in partial differential equations and applying constitutive laws of the isotropic medium are:

$$\tau_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial w}{\partial z} \quad (1)$$

$$\tau_{zz} = (\lambda + 2\mu) \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} \quad (2)$$

$$\tau_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \quad (3)$$

Where τ_{xx} and τ_{zz} are the components in the x and z plane of the strain tensors, the traction forces in the x and z planes, respectively. And τ_{zx} corresponds to the tensile force in the zx plane, where u and w are the displacement components in the x and z axes,

respectively. The variables \mathbf{u} and \mathbf{w} are the velocities of the particles and the Lamé parameters are $\boldsymbol{\rho}$ and $\boldsymbol{\mu}$, where $\boldsymbol{\mu}$ is the stiffness and $\boldsymbol{\rho}$ is the density of the medium.

The step-by-step formulation of a mesh scheme given by Madariaga [2] where he expresses it in terms of particle velocity and tensions, the movement of the elastic wave equation using a circular expansion model in the elastic medium, and Virieux [3] who adapted the general scheme to model waves in a two-dimensional Cartesian system. These schemes facilitate the use of computational simulation. For this Cartesian system the equations that describe movement for the speed of compressional (P) and transverse (S) waves are:

$$\rho \frac{\partial u_t}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad (4)$$

$$\rho \frac{\partial w_t}{\partial t} = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} \quad (5)$$

The compression velocities and shear rate are respectively:

$$\alpha = \mu \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (6)$$

$$\beta = \sqrt{\frac{\mu}{\rho}} \quad (7)$$

Taking the first derivative with respect to time of the equations constituting laws of the medium, and substituting the particle velocity term for displacement, provides a first-order system of velocity and stress equations that can be solved numerically as proposed by Levander [4].

2.1. Propagation in acoustic medium

In seismic exploration, data are acquired through the application of energy pulses, and their subsequent recording in geophone devices, in order to determine the stratigraphy of the subsoil. Seismic modeling is a computational technique that allows us to reconstruct, through simulations, phenomena of energy reflection and refraction in the subsoil. In this way, data acquisition can be simulated, and real records can be compared with synthetic records. In order to carry out the modeling process, it is necessary to specify the expressions for the deformation in an elastic medium, which then allow obtain wave propagation formulas in such medium.

2.2 Propagation in elastic medium

The mathematical model that describes the propagation of vibrations in a rope or acoustic waves in air is called the wave equation, which is classified as a hyperbolic partial differential equation. The speed of the wave is determined by the physical properties of the medium in which it propagates, where $c^2 = F/\rho$, F is the applied tension force and ρ the density of the material. The typical wave equation is shown in expression 8.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (8)$$

Since the medium in which waves propagate in the subsurface is not considered acoustic, the definitions of strain and stress tensors must be taken to model the propagation of energy in an elastic medium. In the previous equations, these tensors and the associated fields have been considered to be in equilibrium. However, when there is a disturbance and waves propagate, the subsurface particles accelerate, gain speed and move. So expression 8 can be reformulated into expression 9.

$$\rho \frac{\partial^2 \bar{u}}{\partial t^2} = \theta_j \tau_{ij} + f_i \quad (9)$$

Knowing the equation of motion for an elastic medium with isotropic properties, we proceed to deduce the wave equation to be solved for a medium in the xz plane. Starting from expression 8 and being $\mathbf{u} = (u_x, u_y, u_z) = (\mathbf{u}, \mathbf{v}, \mathbf{w})$, for the xz plane we have that $\mathbf{u} = (\mathbf{u}, \mathbf{0}, \mathbf{w})$ and the expression in Cartesian coordinates is written as the following way:

$$\rho \frac{\partial^2 u}{\partial t^2} = \partial_j \tau_{xj} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \quad (10)$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \partial_j \tau_{zj} = \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}$$

Applying the definition of τ_{ij} in expression 9, were obtained the definitions for the tensors:

$$\begin{aligned} \tau_{xx} &= (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial w}{\partial z} \\ \tau_{xz} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \tau_{zz} &= (\lambda + 2\mu) \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} \end{aligned} \quad (11)$$

Once these expressions are stated, can be solved using computational methods.

3. COMPUTATIONAL IMPLEMENTATION

A first approach was carried out using a computational environment based on Matlab for Windows, López [5] used a finite difference scheme in an acoustic medium. With this example in mind, a computational implementation scheme is proposed looking for numerical stability properties and trying to eliminate false reflections.

3.1. Discretization using finite differences

The x and z coordinates are discretized by making: $x = (m \pm 1) * h$, $z = (n \pm 1) * h$ and $t = (l \pm 1) * t'$, where h is the interval of the finite difference grid and t' is the time sample in finite differences.

$$\begin{aligned} D_t^+ u_i \left(m, n, l - \frac{1}{2} \right) &= \frac{1}{\rho(m, n)} \left[D_x^- \tau_{xz} \left(m + \frac{1}{2}, n, l \right) + D_x^- \tau_{xz} \left(m, n + \frac{1}{2}, l \right) \right] \\ D_t^+ \tau_{xz} \left(m + \frac{1}{2}, n, l \right) &= \mu \left(m, n + \frac{1}{2}, l \right) \left[D_x^- u_i \left(m, n, l + \frac{1}{2} \right) + D_x^- u_i \left(m + \frac{1}{2}, n + \frac{1}{2}, l + \frac{1}{2} \right) \right] \\ D_t^+ \tau_{xz} \left(m + \frac{1}{2}, n, l \right) &= \mu \left(m, n + \frac{1}{2}, l \right) \left[D_x^+ u_i \left(m, n, l + \frac{1}{2} \right) + D_x^+ u_i \left(m + \frac{1}{2}, n + \frac{1}{2}, l + \frac{1}{2} \right) \right] \end{aligned} \quad (12)$$

Where D_t^+ is the difference operator with respect to time in the subsequent iteration and $D_t^+ \tau_{xz}$ and $D_t^+ u_i$ are the differential operators with respect to space in the subsequent and previous iteration.

3.2. Stability and dispersion properties

For a homogeneous standard medium the spectral analysis is performed in the frequency domain and gives the following stability condition [4]:

$$c_p \frac{\Delta t}{\Delta x} \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta z^2}} < 1 \quad (13)$$

Where c_p is the speed of the P wave. This stability condition is independent of the speed of the S wave. For the special case of a homogeneous mesh the stability condition is:

$$c_p \frac{\Delta t}{\Delta x} < \frac{1}{\sqrt{2}} \quad (14)$$

3.3. Attenuation of false reflections

When using the finite difference method to model the propagation, a problem of false reflections is generated at the edges of the matrix used for the simulation. In a real phenomenon such as when an earthquake occurs, the strong explosion of energy that is generated at such moments causes a wave that continues its path through the subsoil until it loses all its energy. Since the simulation is computational

and finite, it takes place in an area in which the wave transmitting its energy is limited by the size of the mesh (matrix) being used. Artificial borders are generated which cause non-existent reflections that translate into noise or false data. To reduce these false reflections, the scheme described by Cerjan [1] has been implemented, with 20-node wide edges for each border of the mesh. At these edges the propagated value (wave path on the selected edges) is slightly reduced with each time step. The reduction at each selected edge gradually narrows from zero at the inner boundary. Nodes between 1 and 20 on each edge are multiplied by this factor:

$$G = e^{-[0.015 * (20 - i)]^2} \quad (15)$$

Where i takes values between 1 and 20 that correspond to the node in which the wave front is passing. G takes the values of 1 for $i = 20$ and a value of 0.92 for $i = 1$. This scheme can be used in modeling with methods such as finite elements and Fourier analysis, implemented by Pasalic and McGarry [6]. Figure 2 shows an edge absorption scheme at the upper border of the mesh.

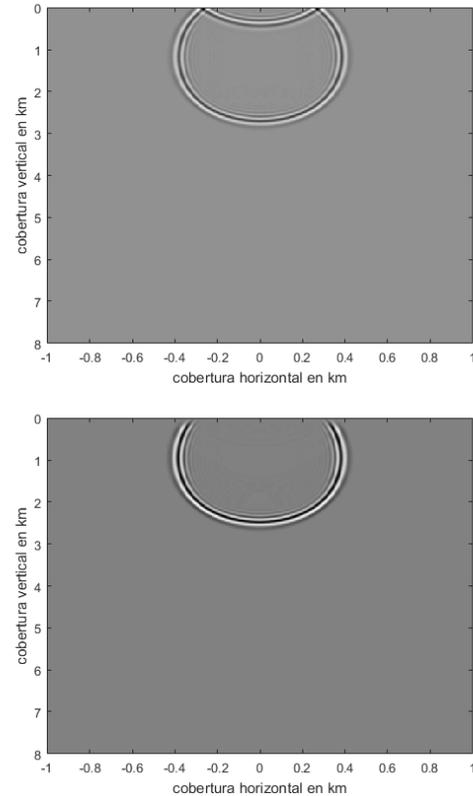


Fig. 2. Comparison of edge-wave reflection reduction scheme. Part "a" shows the wave modeling without attenuation and part "b" shows the absorption attenuation of the left and bottom edges with G coefficient.

4. NUMERICAL EXPERIMENTS

The medium is in equilibrium at time $t = 0$, that is, the velocity and stress components are zero. Propagating the stresses and velocities is equivalent to propagating their components throughout time, Virieux [7]. The first-degree derivatives are discretized using centered finite differences with an eighth-order approximation according to the weights given by the code proposed by Fornberg [8] and are assigned to each node. The main difference with the usual schemes is that the components of the Lamé and medium density coefficients are not known at the same node, as shown in Figure 4. To calculate the square and circular symbols, the Lamé parameters and the density are necessary at time $t\Delta t$ and to calculate the triangular symbols the Lamé coefficients are necessary at time $(t+\frac{1}{2})\Delta t$.

The mesh dimensions are assumed to be 500x500, covering a width of 2km, and a depth of 8 km respectively, at a fundamental frequency of 20 Hz with two parallel plane layers, which originates three regions where the speed is different. The values of Δx and Δz would be 4m and 16m respectively.

Considering flat layers, the first goes from the surface to 2km deep, and a speed of 2000 m/s is assumed; the second goes from 2km to 4km deep assuming a speed of 3000 m/s, the last extends from 4km to 8km with a speed of 4000 m/s.

The computational implementation was carried out on an Intel i5 machine, with 3.3 GHz processor speed and the matlab programming environment was used.

Making some experiments, were simulated two reflectors located at 1000 and 2000 m located below the source, which is 100 m from the surface, were simulated.

5. RESULTS DISCUSSION

Once simulations have been carried out with 800 iterations, results can be seen demonstrating the feasibility of the computational solution to problems of this type. Figures 3 and 4 show the effects of the layers on the reflection and refraction of the propagating wave.

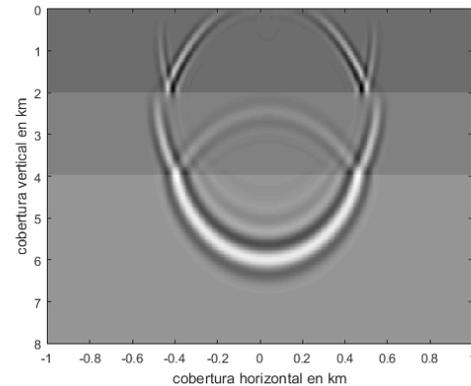


Fig. 3. Wave propagation in two layers of different densities at $t=0.45s$.

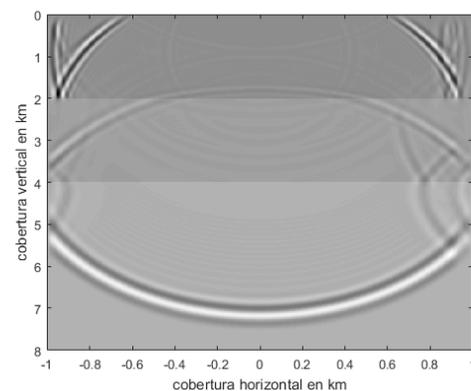


Fig. 4. Wave propagation in two layers of different densities at $t=0.85s$.

Also can be seen the velocity profile considering two reflectors during a simulation time of 0.85s. Here the source is considered to be close to 0 on the horizontal axis.

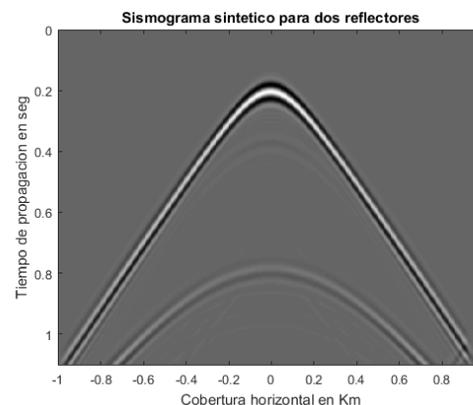


Fig. 5. Velocities profile when considering two reflectors.

As shown in Figure 5, various speeds can be seen, due to the passage of the wave through the various layers, and the reflection of part of it that generates movement in the opposite direction.

Figure 6 shows wave amplitude at a depth of 1800m:

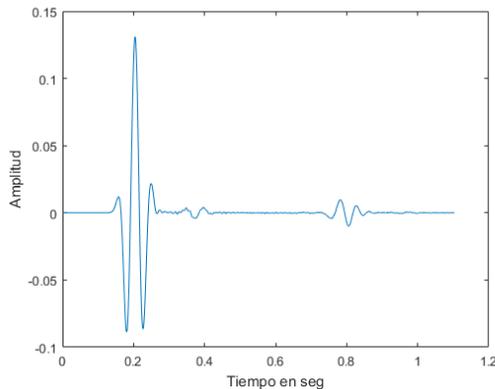


Fig. 6. Wave amplitude during simulation.

As shown, two reflections of the wave are seen at 0.2s and 0.8s due to the passage through different layers, considering that the initial excitation occurred at time 0.

6. CONCLUSIONS

This work carried out demonstrates that it is feasible to simulate the propagation of a wave in a complex geology scheme, through an explicit representation using finite differences. This implementation scheme allows running an approximate simulation for the reality of wave transfer in the elastic medium. In addition, this modeling allows simulating the acquisition of seismic data with several reflectors, as well as with a greater number of layers and other layer arrangements, such as synclines.

The vertical component of the seismogram allows greater visualization of the wave reflections when transmitted in an elastic medium. Due to the behavior of the wave and the various reflections it presents, this simulation is somewhat complicated to understand or visualize clearly, which needs to use schemes that reduce the reflections from the edges since this would further complicate the analysis of the seismogram..

The edge reflection attenuation scheme greatly reduces false reflections at the edges of the mesh even though it does not completely eliminate these reflections and because simulating position of the geophones at the top of the mesh is not great help on the edge.

REFERENCES

- [1] Cerjan C. (1985). "A nonreflecting boundary condition for discrete acoustic and elastic wave equations". *Geophysics*, vol 50, P 705-708.
- [2] Madariaga, R. (1976). "Dynamics of an expanding circular fault, *Bull.*". *Seismic Society Am.*, 66, 639-666.
- [3] Virieux, J. (1984). "SH-Wave propagation in heterogeneous media: velocity-stress finite-difference method". *Geophysics*, vol 49, P 1933-1957.
- [4] Levander, R. (1988). "Fourth-order finite-difference P-SV seismograms". *Geophysics*, vol 53, P 1425-1436.
- [5] López, J. (2017). "Diseño de un algoritmo empleando métodos numéricos para solucionar la ecuación de onda en un medio elástico bidimensional". Universidad de Pamplona. Ingeniería de Sistemas, 2017.
- [6] Pasalic C. y McGarry R. (2010). "Convolutional perfectly matched layer for isotropic and anisotropic acoustic wave equations". 2010 Seg Annual Meeting.
- [7] Virieux, J. (1986). "P-SV wave propagation in heterogeneous media: velocity-stress finite-difference method". *Geophysics*, vol 51 Issue 4, P 889-901.
- [8] Fornberg, B. (1988). "Generation of Finite Difference Formulas on Arbitrarily Spaced Grids". *Mathematics of Computation*, 51, 699-706.