REVISTAS CIENTÍFICAS INDEXADAS BISTUA FACULTAD DE CIENCIAS BÁSICAS - UNIVERSIDAD DE PAMPLONA

***BISTUA** Rev. FCB, Volumen 22 (2) (2024). Pamplona-Colombia.*
https://doi.org/10.24054/bistua.v22i2.2932

Bessel polynomials by context-free grammars

Polinomios de Bessel mediante gramáticas independientes del contexto

Juan Triana[a]

[a] *Universitaria Agustiniana. Bogotá, Colombia. email: juang.triana@uniagustiniana.edu.co*

***Autor Corresponsal:** juang.triana@uniagustiniana.edu.co*

***Recibido**: Mayo 17, 2024.* ***Aceptado**: Junio 23, 2024.* ***Publicado**: Agosto 21, 2024*

Resumen

Los polinomios de Bessel son una familia de polinomios ortogonales $y_n(x)$ presentados como solución de la ecuación diferencial de segundo orden $x^2y'' + 2(x+1)y' = n(n+1)y$; estos polinomios satisfacen la recurrencia $y_n(x) = (2n-1)xy_{n-1}(x) + y_{n-2}(x)$, con $y_1(x) = x+1$ y $y_0(x) = 1$. En términos de la derivada, los polinomios de Bessel pueden obtenerse mediante la recurrencia $y_n(x) = (nx+1)y_{n-1}(x) + x^2y'_{n-1}(x)$. En este artículo estudiamos la conexión entre los polinomios de Bessel y las gramáticas independientes del contexto mediante el operador derivada formal, además demostramos algunas identidades de los polinomios de Bessel.

Palabras clave: Polinomios de Bessel; Operador derivada formal; Gramáticas independientes del contexto.

Abstract:

The Bessel polynomials are an orthogonal family of polynomials $y_n(x)$ introduced as solutions of the second-order differential equation $x^2y'' + 2(x+1)y' = n(n+1)y$, they satisfy the recurrence relation $y_n(x) = (2n-1)xy_{n-1}(x) + y_{n-2}(x)$, where $y_1(x) = x+1$ and $y_0(x) = 1$. In terms of derivatives, Bessel polynomials can be described by the recurrence $y_n(x) = (nx+1)y_{n-1}(x) + x^2y'_{n-1}(x)$. In this paper, we study the connection between Bessel polynomials and context-free grammars through the formal derivative operator, and we prove some identities of this family of polynomials.

Keywords: Bessel polynomials; Formal derivative operator; Context-free grammars.

1. Introduction

Let Σ be an alphabet whose letters are regarded as independent commutative indeterminates. Following [1], a formal function over Σ is defined recursively as follows:

1. Every letter in Σ is a formal function.
2. If u, v are formal functions, then $u+v$ and uv are formal functions.
3. If $f(x)$ is an analytic function, and u is a formal function, then $f(u)$ is a formal function.
4. Every formal function is constructed as above in a finite number of steps.

A context-free grammar G over Σ is defined as a set of substitution rules replacing a letter in Σ by a formal function over Σ.

Definition 1. *Given a context-free grammar G over Σ, the formal derivative operator D, with respect to G, is defined in the following way:*

1. *For u, v formal functions $D(u+v) = D(u) + D(v)$ and $D(uv) = D(u)v + uD(v)$.*
2. *If $f(x)$ is an analytic function and u is a formal function,* $D(f(u)) = \dfrac{\partial f(u)}{\partial u}D(u)$.
3. *For $a \in \Sigma$, if $a \to w$ is a production in G, w being a formal function, then $D(a) = w$; in other cases a is called a constant and $D(a) = 0$.*

For instance, let G be the grammar $G = \left\{a \to ab^2 \,; b \to \dfrac{b^3c}{2} \,; c \to b^2c^2\right\}$, we have $D^0(a) = a$, $D(a) = ab^2$, $D(b) = \frac{b^3c}{2}$, $D(c) = b^2c^2$, $D(c^2) = 2cD(c) = 2b^2c^3$ and $D(ac) = D(a)c + aD(c) = [ab^2]c + a\left[b^2c^2\right] = ab^2c + ab^2c^2$. We next define the iteration of the formal derivative operator.

Definition 2. *For a formal function u, we define $D^{n+1}(u) = D(D^n(u))$ for $n \geq 0$, and $D^0(u) = u$.*

To illustrate the use of Definition 2 we show a connection between context-free grammars and double factorial numbers, which are the numbers defined by the following recurrence relation $n!! = n(n-2)!!$ with $1!! = 1$ and $0!! = 1$. As an interesting fact, double factorial numbers can be expressed in terms of factorial numbers, in the form $(2n)!! = 2^n n!$, and factorial numbers can also be expressed in terms of double factorial numbers: $n! = n!!(n-1)!!$, cf. [14].

Proposition 1. *If* $G = \left\{a \to ab^2\,;b \to \dfrac{b^3c}{2}\,;c \to b^2c^2\right\}$, *then for all* $n \geq 0$ *we have:*

1. $D^n(b^2) = (2n-1)!!b^{2(n+1)}c^n$.
2. $D^n(c) = (2n-1)!!b^{2n}c^{n+1}$.
3. $D^n(c^2) = (2n)!!b^{2n}c^{n+2}$.

Proof. Here we prove 1.; the other results can be similarly proved.

We argue by induction on n. Clearly $D^0(b^2) = b^2$, therefore the proposition is true for $n = 0$. Assuming that $D^n(b^2) = (2n-1)!!b^{2(n+1)}c^n$, $D^{n+1}(b^{2m})$ is calculated as follows:

$$\begin{aligned} D^{n+1}(b^2) &= D(D^n(b^2) = D\left((2n-1))!!b^{2(n+1)}c^n\right) \\ &= (2n-1)!!D(b^{2(n+1)}c^n) \\ &= (2n-1)!!\left[D(b^{2n+2})c^n + b^{2n+2}D(c^n)\right] \end{aligned}$$

Since $D(b^{2n+2}) = (2n+2)b^{2n+1}D(b)$ and $D(c^n) = nc^{n-1}D(c)$, with $D(b) = \frac{b^3c}{2}$ and $D(c) = b^2c^2$, we have that $D^{n+1}(b^2)$ is given by

$$\begin{aligned} &= (2n-1)!!\left[(n+1)b^{2n+4}c^{n+1} + nb^{2n+4}c^{n+1}\right] \\ &= (2n-1)!!(2n+1)b^{2n+4}c^{n+1} \end{aligned}$$

Hence, $D^{n+1}(b^2) = (2n+1)!!b^{2(n+2)}c^{n+1}$. □

The multifactorial numbers $n!_r$ are given by the recurrence relation

$n!_r = n(n-r)!_r$ with $(1-r)!_r = \cdots = (-1)!_r = 0!_r = 1$.

If $r = 1$ we get factorial numbers i.e., $n!_1 = n!$; if $r = 2$ we get double factorial numbers i.e., $n!_2 = n!!$ cf. [14]. The following result can be proven similarly to Proposition 1.

Proposition 2. *If* $G = \left\{a \to ab^2\,;b \to \dfrac{b^3c}{2}\,;c \to b^2c^2\right\}$ *we have:*

1. $D^n(b) = \dfrac{(4n-3)!_4}{2^n}b^{2(n+1)}c^n$.
2. $D^n(bc) = \dfrac{(4n-1)!_4}{2^n}b^{2n+1}c^{n+1}$.

The formal derivative operator of Definition 1 preserves many of the properties of the differential operator in elementary calculus cf. [14], among them the generalized product rule

$$D(u_1 \ldots u_n) = \sum_{k=1}^{n} D(u_k)\prod_{\substack{j=1 \\ j\neq k}}^{n} u_j \tag{1}$$

For instance, if we have the product of three functions we get $D(u_1u_2u_3) = D(u_1)u_2u_3 + u_1D(u_2)u_3 + u_1u_2D(u_3)$. The formal derivative operator also preserves Leibniz's formula

$$D^n(uv) = \sum_{k=0}^{n}\binom{n}{k}D^{n-k}(u)D^k(v). \tag{2}$$

Leibniz's formula is the main tool used in establishing combinatorial properties of the objects generated through context-free grammars [2]. To illustrate the use of Leibniz's formula we present a grammatical proof for the following combinatorial identity.

Proposition 3. $(2n+1)!! = \sum\limits_{k=0}^{n}\binom{n}{k}(2(n-k)-1)!!(2k)!!$ *for all* $n > 0$.

Proof. By Proposition 1, $D^{n+1}(c) = (2n+1)!!b^{2(n+1)}c^{n+2}$. On the other hand, by Definition 2 we have $D^{n+1}(c) = D^n(D(c)) = D^n(b^2c^2)$. By Leibniz's formula we get

$$D^n(b^2c^2) = \sum_{k=0}^{n} D^{n-k}(b^2)D^k(c^2). \tag{3}$$

Since $D^{n-k}(b^2) = (2(n-k)-1)!!b^{2(n-k+1)}c^{n-k}$ and $D^k(c^2) = (2k)!!b^{2k}c^{k+2}$, by Proposition 1, we obtain that $D^n(b^2c^2)$ is equal to

$$\begin{aligned} &\sum_{k=0}^{n}[(2(n-k)-1)!!b^{2(n-k+1)}c^{n-k}][(2k)!!b^{2k}c^{k+2}] \\ &= \sum_{k=0}^{n}(2(n-k)-1)!!(2k)!!b^{2n+2}c^{n+2} \end{aligned}$$

Thus we obtain

$$\begin{aligned} D^{n+1}(c) &= D^n(b^2c^2) \\ (2n+1)!!b^{2n+2}c^{n+2} &= \sum_{k=0}^{n}(2(n-k)-1)!!(2k)!!b^{2n+2}c^{n+2}. \end{aligned}$$

By equating the coefficients of $b^{2n+2}c^{n+2}$ we conclude that $(2n+1)!! = \sum\limits_{k=0}^{n}(2(n-k)-1)!!(2k)!!$ □

Since $D^n(c^2) = D^n(cc)$ by Leibniz's formula and Proposition 1 we may deduce that

$$(2n)!! = \sum_{k=0}^{n}(2(n-k)-1)!!(2k-1)!! \tag{4}$$

A combinatorial proof of the Equation (4) can be found in [5]. On the other hand, by applying Leibniz formula on $D^n(bc)$ we may deduce

$$\frac{(4n-1)!_4}{2^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{(4(n-k)-3)!_4}{2^{n-k}}(2k-1)!! \tag{5}$$

The formal derivative operator defined with respect to context-free grammars has been used to study combinatorial objects cf. [6,10,11,17], families of numbers cf. [14,15], families of polynomials cf. [3,9] among others. In [13], the formal derivative operator defined with respect to matrix grammars and a connection with multifactorial numbers were introduced.

In this paper, we present the relation between Bessel polynomials and context-free grammars, we introduce the grammar $G=\left\{a\rightarrow ab^2\,;b\rightarrow \dfrac{b^3c}{2}\,;c\rightarrow b^2c^2\right\}$ and we use it to prove some properties.

2. Context-free grammars and Bessel polynomials

The Bessel polynomials $y_n(x)$ were introduced in [7], they can be defined as the polynomial solution of the second-order differential equation

$$x^2\frac{d^2y_n}{dx^2}+2(x+1)\frac{dy_n}{dx}=n(n+1)y_n(x). \tag{6}$$

Since $y_0(x)=1$ and $y_1(x)=x+1$, $y_n(x)$ can be easily obtained via the following recurrence relation

$$y_n(x)=(2n-1)xy_{n-1}(x)+y_{n-2}(x). \tag{7}$$

In terms of derivatives $y_n(x)$ satisfies the recurrence relation

$$y_n(x)=(nx+1)y_{n-1}(x)+x^2y'_{n-1}(x). \tag{8}$$

The first Bessel polynomials are:

$$\begin{aligned}
y_0(x)&=1\\
y_1(x)&=x+1\\
y_2(x)&=3x^2+3x+1\\
y_3(x)&=15x^3+15x^2+6x+1\\
y_4(x)&=105x^4+105x^3+45x^2+10x+1\\
y_5(x)&=945x^5+945x^4+420x^3+105x^2+15x+1
\end{aligned}$$

The following result shows the relation between Bessel polynomials and the context-free grammar $G=\left\{a\rightarrow ab^2\,;b\rightarrow \dfrac{b^3c}{2}\,;c\rightarrow b^2c^2\right\}$.

Proposition 4. *If* $G=\left\{a\rightarrow ab^2\,;b\rightarrow \dfrac{b^3c}{2}\,;c\rightarrow b^2c^2\right\}$ *then:*

1. $D^n(a)=ab^{2n}y_{n-1}(c)$ *for all* $n\geq 1$.
2. $D^n(ac)=ab^{2n}cy_n(c)$ *for all* $n\geq 0$.

Proof. Here we prove 1., the result 2. can be similarly proved.

We argue by induction on n. Since $D(a)=ab^2=ab^{2(1)}y_0(c)$, the result is true for $n=1$. Assuming that $D^n(a)=ab^{2n}y_{n-1}(c)$, we calculate $D^{n+1}(a)$ as follows

$$\begin{aligned}
&D(D^n(a))\\
z&=D(ab^{2n}y_{n-1}(c))\\
&=D(a)b^{2n}y_{n-1}(c)+aD(b^{2n})y_{n-1}(c)+ab^{2n}D(y_{n-1}(c))
\end{aligned}$$

By Definition 1 we get

$$ab^{2n+2}y_{n-1}(c)+a\left[2nb^{2n-1}D(b)y_{n-1}(c)+b^{2n}y'_{n-1}(c)D(c)\right]$$

Since $D(a)=ab^2,\ D(b)=\dfrac{b^3c}{2}$ and $D(c)=b^2c^2$ we obtain that $D^{n+1}(a)$ is given by

$$\begin{aligned}
&ab^{2n+2}y_{n-1}(c)+a\left[nb^{2n+2}cy_{n-1}(c)+b^{2n+2}c^2y'_{n-1}(c)\right]\\
&=ab^{2(n+1)}\left[(1+nc)y_{n-1}(c)+c^2y'_{n-1}(c)\right].
\end{aligned}$$

Thus, by Equation (8), we get $D^{n+1}(a)=ab^{2(n+1)}y_n(c)$. □

An explicit formula for $y_n(x)$, known as Rodrigues' formula [4], is given by

$$y_n(x)=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k \tag{9}$$

Therefore, the Bessel polynomials $y_n(x)$ can be expressed as $\sum_{k=0}^{n}B_{n,k}x^k$ where $B_{n,k}=\frac{(n+k)!}{2^k(n-k)!k!}$ for $0\leq k\leq n$; in other cases we define $B_{n,k}=0$. The coefficients of Bessel polynomials $B_{n,k}$ can be found as the triangular array A001498 in OEIS (the On-line Encyclopedia of Integer Sequences).

n / k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	3	3			
3	1	6	15	15		
4	1	10	45	105	105	
5	1	15	105	420	945	945

Tabla 1: The first $B_{n,k}$ coefficients.

The following proposition is a recurrence relation to generate the triangular array of $B_{n,k}$ numbers

Proposition 5. $B_{n+1,k}=(n+k)B_{n,k-1}+B_{n,k}$

Proof. Since $y_{n+1}(x)=[(n+1)x+1]y_n(x)+x^2y'_n(x)$, we have that $B_{n+1,k}x^k$ is given by

$$\begin{aligned}
&[(n+1)xB_{n,k-1}x^{k-1}+B_{n,k}x^k]+x^2(k-1)B_{n,k-1}x^{k-2}\\
&=(n+k)B_{n,k-1}x^k+B_{n,k}x^k
\end{aligned}$$

Thus, by equating the coefficients of x^k we conclude that $B_{n+1,k}=(n+k)B_{n,k-1}+B_{n,k}$ □

It is easy to prove that $B_{n,0}=1$ for each $n>0$, from Proposition 5 considering $k=0$. Similarly, considering $k=n$ we may obtain a connection between double factorial numbers and the diagonal of the triangular array presented in Table 1.

Corollary 1. $B_{n,n}=(2n-1)!!$ *and* $B_{n,n}=B_{n,n-1}$ *for each* $n\geq 1$.

The Bessel polynomials can be generated through several context-free grammars. For instance, if $G = \{a \to a^3; b \to ab\}$ it can be verified that $D^n(ab) = a^{n+1}by_n(a)$ cf. [6]. On the other hand, for the grammar $\{a \to ab + ab^2; b \to b^3\}$ we get $D^n(a) = ab^n y_n(b)$ cf. [11]. If $\{a \to ab\ ; b \to b^2c\ ; c \to bc^2\}$, then $D^n(ab) = ab^{n+1}y_n(c)$ and $D^n(a^2b) = 2^n a^2 b^{n+1} y_n\left(\frac{c}{2}\right)$ cf. [8], in addition for this grammar we may to prove that $D^n(ac) = ab^n c y_n(c)$. In [16], was introduced the grammar $G = \{a \to ax + ay; x \to 2x^2; y \to xy\}$ such that $D^n(a) = a\sum_{k=0}^{n} B_{n,k}x^k y^{n-k}$.

The results above can be proved by induction via the recurrence relation for Bessel polynomials given in Equation (8), similar to Proposition 4.

3. Some identities of Bessel polynomials

The following proposition was presented in [8] by the grammar $\{a \to ab\ ; b \to b^2c\ ; c \to bc^2\}$. Here we give a proof using the context-free grammar $G = \left\{a \to ab^2\ ; b \to \frac{b^3c}{2}\ ; c \to b^2c^2\right\}$.

Proposition 6. $y_n(x) = \sum_{k=0}^{n} \binom{n}{k}(2n-2k-1)!!y_{k-1}(x)x^{n-k}$.

Proof. Let G be the grammar $\left\{a \to ab^2\ ; b \to \frac{b^3c}{2}\ ; c \to b^2c^2\right\}$. By Proposition 4, $D^{n+1}(a) = ab^{2n+2}y_n(c)$; since $D^{n+1}(a) = D^n(D(a)) = D^n(ab^2)$, we get

$$ab^{2n+2}y_n(c) = D^n(ab^2). \tag{10}$$

By Leibniz formula on $D^n(ab^2)$ we obtain

$$ab^{2n+2}y_n(c) = \sum_{k=0}^{n}\binom{n}{k}D^k(a)D^{n-k}(b^2). \tag{11}$$

By Proposition 1, $D^n(b^2) = (2n-1)!!b^{2(n+1)}c^n$; replacing in (11), we have that $ab^{2n+2}y_n(c)$ is given by

$$\sum_{k=0}^{n}\binom{n}{k}ab^{2k}y_{k-1}(c)(2(n-k)-1)!!b^{2(n-k+1)}c^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n}{k}(2n-2k-1)!!y_{k-1}(c)c^{n-k}ab^{2n+2}.$$

Since both sides of the above equality have ab^{2n+2}, we conclude

$$y_n(x) = \sum_{k=0}^{n}\binom{n}{k}(2n-2k-1)!!y_{k-1}(x)x^{n-k}.$$

□

Proposition 6 can be similarly proved by using Leibniz's formula on $D^n(ac)$. The following proposition shows that all the roots of Bessel polynomials are simple roots.

Proposition 7. *If r is a root of $y_n(x)$, r is a simple root.*

Proof. We argue by contradiction. Assuming that r is a non-simple root of $y_n(x)$ we have that $y_n(r) = 0$ and $y_n'(r) = 0$, therefore

$$y_{n+1}(r) = [(n+1)r + 1]y_n(r) + r^2y_n'(r) = 0. \tag{12}$$

On the other hand, by equation (7) and taking into account that $y_n(r) = y_{n+1}(r) = 0$ we get

$$y_{n+1}(r) = [2(n+1) - 1]ry_n(r) + y_{n-1}(r)$$
$$0 = 0 + y_{n-1}(r)$$

Thus $y_{n-1}(r) = 0$. Since $y_n(r) = [nr + 1]y_{n-1}(r) + r^2y_{n-1}'(r)$ and $y_{n-1}(r) = y_n(r) = 0$ we get $r^2y_{n-1}'(r) = 0$, therefore $r = 0$ or $y_{n-1}'(r) = 0$. Since $B_{n,0} = 1$ for each n we have that $r \neq 0$, thus $y_{n-1}'(r) = 0$. Hence for each n we have that if r is a non-simple root of $y_n(x)$ then is a non-simple root of $y_{n-1}(x)$; furthermore, we may conclude that r is a non-simple root of $y_{n-2}(x)$ so we have a contradiction because if we continue with that reasoning if r is a non-simple root of $y_n(x)$ it will be a non-simple root of $y_2(x)$ but roots of $y_2(x)$ are complex conjugate simple roots.

Hence all the roots of $y_n(x)$ are simple roots. □

Since $0 \leq B_{n,1} \leq \cdots \leq B_{n,n}$ for each n and $B_{n,k} \in \mathbb{R}$ for each n and k, by Eneström-Kakeya theorem we have that all the zeros of $y_n(z)$ lies on $|z| < 1$ cf. [12], therefore we may think that the Bessel polynomials must have common roots.

Corollary 2. *$y_{n+1}(x)$ and $y_n(x)$ do not have common roots.*

Proof. We argue by contradiction. Assuming that $y_n(r) = y_{n+1}(r) = 0$ we have

$$y_{n+1}(r) = [(n+1)r + 1]y_n(r) + r^2y_n'(r)$$
$$0 = 0 + r^2y_n'(r)$$

Since $B_{n,0} = 1$ for each n we have that $r \neq 0$, therefore $y_{n-1}'(r) = 0$. Hence r is a non-simple root of $y_{n-1}(x)$, but by Proposition 7 we obtain that $y_{n-1}(x)$ does not have non-simple roots, thus we have a contradiction. □

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Acknowledgement

The author would like to thank the referees for their constructive comments and suggestions.

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