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Approximation and Simulation of Signal Belongs to Generalized Weighted Lipschitz Class by $(N, p_m, q_m)(C, \alpha, \eta)(E, \theta)$ Product Transform

Aproximación y Simulación de Señal Perteneciente a la Clase de Lipschitz Generalizada Ponderada $(N, p_m, q_m)(C, \alpha, \eta)(E, \theta)$

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Resumen

En este estudio, la conocida serie conjugada de Fourier, tradicionalmente sumable mediante varios métodos individuales, demuestra una mayor velocidad de convergencia y una mejor aproximación de señales al aplicarse una transformación de producto. Este trabajo tiene como objetivo establecer un nuevo teorema para aproximar señales dentro de una clase específica de funciones, utilizando la sumabilidad en producto de series conjugadas de Fourier. Además, se presentan una serie de ejemplos ilustrativos para validar el método de sumabilidad propuesto y destacar su comportamiento de convergencia. Los resultados se respaldan mediante simulaciones realizadas en programación MATLAB.

Palabras clave: Lipschitz generalizada ponderada; Serie de Fourier conjugada; Desigualdad de Hölder

Abstract:

In this study, the well-known Fourier Conjugate series, which is traditionally summable through various individual methods, demonstrates enhanced convergence speed and improved signal approximation when subjected to a product transform. This work aims to establish a novel theorem for approximating signals within a specific function class, utilizing product summability of conjugate Fourier series. Additionally, a range of illustrative examples is provided to validate the proposed summability method and highlight their convergence behavior. The findings are further supported through simulations conducted using MATLAB programming.

Keywords: Generalized Weighted Lipschitz; Conjugate Fourier Series; Hölder Inequality

1. Introduction and Motivation

Summability theory has various applications and plays an important role for study of functional analysis. Weierstrass theorem was used in starting to originate the approximation theory and further this study was carried out using trigonometric polynomials. Signal approximation is crucial because it transmits information or characteristics about a phenomenon. Engineers and scientists used Fourier approximation features to create finite impulse response (FIR) digital filters with enhanced performance. Product operators have applications in signal theory, mechanical engineering, machine theory and digital filter design. Zygmund [3] proposed the trigonometric approximation of signals for the periodic series. Several researchers studied the approximation of signals or functions by various summability methods like Cesàro, Euler, Riesz and Nörlund-mean etc. Rhoades et. al [5], Nigam [11], Mittal et. al [16], [17], Sonker and Singh [19], Mishra et. al [20] carried out their study on approximation of signals belong to Lipschitz classes by linear operators. Bor [7], [8] generalized the known results by using a class of infinite and Fourier series. An infinite series that cannot be summable by the left or right linear operators independently can be summable to a number using product operators, which made the product operators advantageous over linear operators. Thakur et. al. [1], Rathore and Singh [2], Khan [10], Qureshi [13], Mittal et. al [15], Deger [22], Singh [23], Singh and Srivastava [24] have proved theorems on approximation of functions belong to weighted class. Lal and Nigam [14] worked on approximation of signal by matrix summability. As a result, the summability means' subsequences are usually always going to converge. Recently, Sonker and Jindal [18] studied triple product summability for the better approximation. Rathore and Shrivastava [12] worked on Euler-Nörlund product means. Krasniqi [21] worked on the approximation of functions belong to Lip class by Cesàro-Euler means. However, nothing appears till to obtain the approximation of signals belong to $W^{'}\left(L^{p},\xi\left(t\right)\right),\left(p\geq1\right),\left(t>0\right)$ using $\left(N,p_{m},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ product operator of conjugate Fourier series. In comparison to the another established product means, the proposed product transform can potentially achieve faster convergence rates and more accurate approximations, especially for functions that are difficult to approximate using traditional methods. Improved error estimates, Robustness against noise and Flexibility in function classes are another features of the proposed methods.

(I). Let $\sum u_{m}$ be a given infinite series and $\{s_{m}\}$ be the sequence of its m^{th} partial sums. Let $p=\{p_{m}\}$ be a non-increasing, monotonic and +ive sequence such that

$$P_{m}=\sum_{w=0}^{m}p_{w}\rightarrow\infty,\ as\,m\rightarrow\infty. \tag{1..1}$$

$$P_{-1}=p_{-1}=0,\ \forall\,i\geq1.$$

For sequence $q=\{q_{m}\}$, we define an increasing sequence $\{R_{m}\}$ as

$$R_{m}=\left(p*q\right)_{m}=\sum_{w=0}^{m}p_{m-w}q_{w}\rightarrow\infty,\ as\,m\rightarrow\infty \tag{1..2}$$

represents the convolution product in which

$$Q_{m}=\sum_{w=0}^{m}q_{w}\rightarrow\infty,\ Q_{-1}=q_{-1}=0,\ \forall\,i\geq1. \tag{1..3}$$

(II). The transformation (sequence -to -sequence)

$$t_{m}^{N}=\frac{1}{R_{m}}\sum_{w=0}^{m}p_{m-w}q_{w}s_{w} \tag{1..4}$$

gives the sequence $\{t_{m}\}$ of the $\left(N,p_{m},q_{m}\right)$ mean of $\{s_{m}\}$ (Borwein [4]). The infinite series $\sum u_{m}$ is $\left(N,p_{m},q_{m}\right)$ summable to s, if $t_{m}^{N}\rightarrow s$, as $m\rightarrow\infty$.

(III).

$$If\ C_{m}^{(\alpha,\eta)}=\frac{1}{A_{m}^{\alpha+\eta}}\sum_{h=0}^{m}A_{m-h}^{\alpha-1}\ A_{h}^{\eta}s_{h}\rightarrow \text{s} \quad \text{as m}\rightarrow\infty, \tag{1..5}$$

the m^{th} Cesàro means (α,η) with $\alpha+\eta>-1$ of $\{s_{m}\}$, *i.e.* (see [6]) where $A_{m}^{\alpha+\eta}=O\left(m^{\alpha+\eta}\right),\alpha+\eta>-1\ and\ A_{0}^{\alpha+\eta}=1$, then $\sum u_{m}$ is Cesàro (C,α,η) summable to 's' and denoted as $C_{m}^{(\alpha,\eta)}$.

(IV).

$$If\ \ E_{m}^{\theta}=\frac{1}{\left(1+\theta\right)^{m}}\sum_{h=0}^{m}\binom{m}{h}\theta^{m-h}s_{h}\rightarrow \text{s} \quad \text{as m}\rightarrow\infty, \tag{1..6}$$

then $\sum u_{m}$ is Euler (E,θ) summable to 's'.

(V). The product of (N,p_{m},q_{m}) with (C,α,η) (E,θ) gives (N,p_{m},q_{m}) (C,α,η) (E,θ) summabilty and represented by $(NCE)_{m}^{p,q;\alpha,\eta;\theta}$.

$$\begin{aligned} If\ \ (NCE_{m}^{p,q;\alpha,\eta;\theta}&=\frac{1}{R_{m}}\sum_{h=0}^{m}p_{m-h}q_{h}\frac{1}{A_{h}^{\alpha+\eta}}\\ &\times\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_{i}^{\eta}E_{i}^{\theta}\rightarrow s\quad as\,m\rightarrow\infty\\ &=\frac{1}{R_{m}}\sum_{h=0}^{m}p_{m-h}q_{h}\frac{1}{A_{h}^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_{i}^{\eta}\frac{1}{\left(1+\theta\right)^{i}}\\ &\times\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\rightarrow s\quad as\,m\rightarrow\infty, \end{aligned} \tag{1..7}$$

then the series $\sum u_{m}$ is $\left(N,p_{m},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ summable to the definite number 's'.

(VI). Let $s_{m}\rightarrow s$ implies $E_{m}^{\theta}\left(s_{m}\right)\rightarrow s$ as $m\rightarrow\infty$. Hence (E,θ) method is regular. Now we may write, $(NCE)_{m}^{p,q;\alpha,\eta;\theta}=(NC)_{m}^{p,q;\alpha,\eta}\left(E_{m}^{\theta}\left(s_{m}\right)\right)\rightarrow s$ as $m\rightarrow\infty$. Therefore $\left(N,p_{m},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ method is also regular.

Remark: If we put q_{m} = 1 for all $m\in N$, then $\left(N,p_{m},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ method reduces to (N,p_{m}) $\left(C,\alpha,\eta\right)\left(E,\theta\right)$ summability method and if we put p_{m} = 1 for all $m\in N$, then $\left(N,p_{m},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ reduces to $\left(\overline{N},q_{m}\right)\left(C,\alpha,\eta\right)\left(E,\theta\right)$ method.

(VII). Let a signal denoted by g having 2π as periodic time and is integrable in the same way as of Lebesgue for the limit $(-\pi,\pi)$. Let

$$s_{m}\left(g;x\right)=\frac{a_{0}}{2}+\sum_{h=1}^{m}\left(a_{h}\cos hx\right)+\sum_{h=1}^{m}\left(b_{h}\sin hx\right) \tag{1..8}$$

be the Fourier series and

$$\sum_{h=1}^{\infty}\left(b_{h}\cos hx\right)-\sum_{h=1}^{\infty}\left(a_{h}\sin hx\right) \tag{1..9}$$

be its conjugate series and m^{th} partial sum of (1..9)is

$$\overline{s_{m}}\left(g;x\right)=\sum_{h=1}^{m}\left(b_{h}\cos hx\right)-\sum_{h=1}^{m}\left(a_{h}\sin hx\right) \tag{1..10}$$

Definition 1: For g, the L_{∞} - norm is represented as $\|g\|_{\infty}$ and is given by

$$\|g\|_{\infty}=\sup\left\{|g\left(x\right)|:x\in R\right\} \tag{1..11}$$

whereas L_{p} - norm, is represented as $\|g\|_{p}$, defined for $[0,2\pi]$ and is given by

$$\|g\|_{p}=\left\{\int_{0}^{2\pi}|g\left(x\right)|^{p}dx\right\}^{1/p},\ \text{p}\geq1. \tag{1..12}$$

Definition 2: The approximation of g by $t_{m}\left(x\right)$ under $\|.\|_{\infty}$

is given by Zygmund [3] with

$$\|t_m - g\|_\infty = \sup\{|t_m(x) - g(x)| : x \in R\} \quad (1..13)$$

and the Trigonometric approximation $E_m(g)$ of $g \in L_p$ is defined as

$$E_m(g) = \min_m \|t_m(g;x) - g(x)\|_p \quad (1..14)$$

Definition 3: A real valued signal g is of Lipschitz class usually denoted as $g \in Lip\beta$ if

$$|g(x+t) - g(x)| = O\left(|t|^\beta\right),\ 0 < \beta \le 1, t > 0 \quad (1..15)$$

and $g \in Lip(\beta, p)$ if

$$\omega_p(t;g) = \left(\int_0^{2\pi} |g(x+t) - g(x)|^p dx\right)^{\frac{1}{p}} = O\left(|t|^\beta\right)\ for\ 0 < \beta \le 1,\ p \ge 1,\ t > 0. \quad (1..16)$$

For $\xi(t)$ (+ve increasing signal) and $p \ge 1$, $g \in Lip\ (\xi(t), p)$ if

$$\omega_p(t;g) = \left(\int_0^{2\pi} |g(x+t) - g(x)|^p dx\right)^{\frac{1}{p}} = O(\xi(t))\ for\ p \ge 1,\ t > 0. \quad (1..17)$$

For $\xi(t)$ and $p \ge 1$, a real valued signal $g \in W'(L^p, \xi(t))$, ([10]), if

$$\omega_p(t;g) = \left(\int_0^{2\pi} |g(x+t) - g(x)|^p \sin^{\gamma p}(x)\, dx\right)^{\frac{1}{p}} = O(\xi(t))\ for\ \gamma \ge 0,\ p \ge 1,\ t > 0. \quad (1..18)$$

We redefine the weighted class ([5], [9]) to evaluate $I_{1.2}$ without error and gives as

$$\omega_p(t;g) = \left(\int_0^{2\pi} |g(x+t) - g(x)|^p \sin^{\gamma p}\left(\frac{x}{2}\right) dx\right)^{\frac{1}{p}} = O(\xi(t))\ for\ \gamma \ge 0,\ t > 0. \quad (1..19)$$

If $\gamma = 0$, then $W'(L^p, \xi(t))$ coincides with $Lip\ (\xi(t), p)$, and

$$W'(L^p, \xi(t)) \xrightarrow{\gamma=0} Lip\ (\xi(t), \mathrm{p}) \quad (1..20)$$

$$\xrightarrow{\xi(t)=t^\beta} Lip(\beta, p) \xrightarrow{p\to\infty} Lip\beta$$
$$for\ 0 < \beta \le 1, p \ge 1,\ t > 0. \quad (1..21)$$

Notations:

$$\psi_x(t) = g(x+t) - g(x-t)$$

$$\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t) = \frac{1}{R_m}\sum_{h=0}^{m} p_{m-h} q_h \frac{1}{A_h^{\alpha+\eta}} \sum_{i=0}^{h}\left[A_{h-i}^{\alpha-1}\ A_i^\eta \frac{1}{(1+\theta)^i} \times \left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{\cos\left(j+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right\}\right] \quad (1..22)$$

and $\tau = \left[\frac{1}{t}\right]$, the integral part of $\frac{1}{t}$.

2. Known Results

Dealing with (E, q) (C, θ, β) mean, Krasniqi [21] have proved the following theorems for a conjugate function $\overline{g}$:

Theorem 2..0.1. *If $\overline{g}$ belongs to the Lip α class, then its approximation by $(E,q)\ (C,\theta,\beta)$ means of (1..10) is*

$$\sup_{0<x<2\pi} |\overline{(EC)_m^{q;\theta,\beta}}(\overline{g}(x)) - \overline{g}(x)| = \|\overline{(EC)_m^{q;\theta,\beta}}(\overline{g}) - \overline{g}\|_\infty = O\left(\frac{1}{(m+1)^\alpha}\right) \quad (2..1)$$

$$,\ 0 < \alpha < 1. \quad (2..2)$$

Theorem 2..0.2. *If $\overline{g}$ belongs to the $W(L_p, \xi(t))$ class, then approximation by $(E,q)\ (C,\theta,\beta)$ means of (1..10) is given by*

$$\|\overline{(EC)_m^{q;\theta,\beta}}(\overline{g}) - \overline{g}\|_p = O\left((m+1)^{\gamma+\frac{1}{p}}\ \xi\left(\frac{1}{m+1}\right)\right) \quad (2..3)$$

provided

$$\left\{\int_0^{\frac{1}{m+1}}\left(\frac{|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\gamma p}\frac{t}{2} dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{(m+1)^{\frac{1}{p}}}\right), \quad (2..4)$$

and

$$\left\{\int_{\frac{1}{m+1}}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left((m+1)^\delta\right) \quad (2..5)$$

For δ (arbitrary number), $q(1-\delta) - 1 > 0, \frac{1}{p} + \frac{1}{q} = 1, 1 \le p < \infty$.
The conditions(2..4) and (2..5) are always true in x and

$$\overline{g}(x) = -\frac{1}{2\pi}\int_0^\pi \phi_x(t)\cot\left(\frac{t}{2}\right) dt. \quad (2..6)$$

Dealing with (E, q) (N, p_m) product mean, Rathore and Shrivastava [12] have proved:

Theorem 2..0.3. *If g belongs to the $W(L_p, \xi(t))$ class, then approximation by (E, q) (N, p_m) product mean (1..8) is given* by

$$\|(EN)_m^{q;p}(g)-g\|_p = O\left((m+1)^{\gamma+\frac{1}{p}}\xi\left(\frac{1}{m+1}\right)\right) \quad (2..7)$$

provided

$$\left\{\int_0^{\frac{\pi}{m+1}}\left(\frac{t|\phi_x(t)|}{\xi(t)}\right)^p \sin^{\gamma p} t dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{(m+1)}\right), \quad (2..8)$$

and

$$\left\{\int_{\frac{\pi}{m+1}}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|\sin^{\gamma} t}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left((m+1)^{\delta}\right) \quad (2..9)$$

For δ *(arbitrary number),* $q(1-\delta)-1>0, \frac{1}{p}+\frac{1}{q}=1, 1\le p<\infty$.
The conditions(2..8) and (2..9) are always true in x.

3. Lemma

Lemma 3..0.1. $|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| = O\left[\frac{1}{t}\right]$, *for* $0<t\le\frac{\pi}{(1+m)}$*;* $t\le\pi\sin\left(\frac{t}{2}\right)$ *and* $|\cos(mt)|\le 1$.
Proof:

$$\begin{aligned}
|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| &\le \frac{1}{2\pi.R_m}\left|\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{\cos\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|\\
&\le \frac{1}{2\pi.R_m}\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\\
&\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{|\cos\left(j+\frac{1}{2}\right)t|}{|\sin\frac{t}{2}|}\right\}\\
&\le \frac{1}{2\pi.R_m}\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\\
&\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{1}{\frac{t}{\pi}}\right\}\\
&= \frac{1}{2t.R_m}\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\\
&\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\right\}\\
&= \frac{1}{2t.R_m}\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta} \quad (3..1)\\
&\left\{\because\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}=(1+\theta)^i\right\}\\
&= \frac{1}{2t.R_m}\sum_{h=0}^{m} p_{m-h}q_h\left\{\because\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}=A_h^{\alpha+\eta}\right\}\\
&= O\left[\frac{1}{t}\right].
\end{aligned}$$

Lemma 3..0.2. $|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| = O\left[\frac{1}{t}\right]$, *for* $0<\frac{\pi}{(1+m)}\le t\le\pi$*;* $t\le\pi\sin\left(\frac{t}{2}\right)$.
Proof:

$$\begin{aligned}
|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| &\le \frac{1}{2\pi.R_m}\left|\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{\cos\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|\\
&\le \frac{1}{2\pi.R_m}\left|\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{\cos\left(j+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right\}\right|\\
&\le \frac{1}{2t.R_m}\left|\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{i\left(j+\frac{1}{2}\right)t}\right\}\right|\\
&\le \frac{1}{2t.R_m}\left|\sum_{h=0}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{ijt}\right\}\right||e^{i\frac{t}{2}}|\\
&\le \frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{ijt}\right\}\right|\\
&+\frac{1}{2t.R_m}\left|\sum_{h=\tau}^{m} p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h} A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.\\
&\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{ijt}\right\}\right|. \quad (3..2)
\end{aligned}$$

Now, consider first term of (3..2), we have

$$\frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.$$
$$\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{ijt}\right\}\right|$$
$$\leq\frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.$$
$$\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\right\}||e^{ijt}\right|$$
$$\leq\frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\right.$$
$$\left.\frac{1}{(1+\theta)^i}\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\right.$$
$$=\frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\right|$$
$$=\frac{1}{2t.R_m}\left|\sum_{h=0}^{\tau-1}p_{m-h}q_h\right|\quad\left\{\because\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}=A_h^{\alpha+\eta}\right\}$$
$$=O\left[\frac{1}{t}\right]. \tag{3..3}$$

Using Able's Lemma and considering second term of (3..2)

$$\frac{1}{2t.R_m}\left|\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}\right.$$
$$\left.\times Re\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}e^{ijt}\right\}\right|$$
$$\leq\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}$$
$$\times\max 0\leq k\leq i\left|\sum_{j=0}^{k}\binom{i}{j}\theta^{i-j}e^{ijt}\right|$$
$$\leq\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}$$
$$\times\max 0\leq k\leq i\sum_{j=0}^{k}\binom{i}{j}\theta^{i-j}|e^{ijt}|$$
$$=\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}$$
$$\times\max 0\leq k\leq i\sum_{j=0}^{k}\binom{i}{j}\theta^{i-j}$$
$$=\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}$$
$$\times\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}$$
$$=\frac{1}{2t.R_m}\sum_{h=\tau}^{m}p_{m-h}q_h\quad\left\{\because\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}=A_h^{\alpha+\eta}\right\}$$
$$=O\left[\frac{1}{t}\right] \tag{3..4}$$

From (3..3) and (3..4),

$$\left|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)\right|=O\left[\frac{1}{t}\right].$$

4. Main Theorem

Theorem 4..0.1. *If a signal $\overline{g}$ with time period of 2π, integrable as Lebesgue for $(-\pi,\pi)$ and of class $W'(L^p,\xi(t)),(p\geq1),(t>0)$, then the approximation by $\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}$ product means of (1..10) is given by*

$$\|t_m^{\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}}(g;x)-\overline{g}\|_p=O\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right). \tag{4..1}$$

If $\left\{\xi(t).t^{-1}\right\}$ is a non-increasing sequence,

$$\left(\int_0^{\frac{\pi}{(1+m)}}\left(\frac{|\psi_x(t)|}{\xi(t)}\right)^p\sin^{\gamma p}\left(\frac{t}{2}\right)dt\right)^{\frac{1}{p}}=O(1) \tag{4..2}$$

$$\left(\int_{\frac{\pi}{(1+m)}}^{\pi}\left(\frac{|\psi_x(t)|}{\xi(t).t^{\delta}}\right)^p dt\right)^{\frac{1}{p}}=O\left(\frac{1}{(1+m)^{-\delta}}\right) \tag{4..3}$$

For δ (arbitrary number), $(1-\delta)q-1>0,\frac{1}{p}+\frac{1}{q}=1,1\leq p\leq\infty$.
The conditions(4..2) and (4..3) are always true in x and $\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}$ is $\overline{(N,p_m,q_m)(C,\alpha,\eta)(E,\theta)}$ summable of (1..10) and $\overline{g}(x)$ is defined by

$$2\pi\overline{g}(x)=-\lim_{\epsilon\to0}\int_{\epsilon}^{\pi}\psi_x(t)\cot\left(\frac{t}{2}\right)dt. \tag{4..4}$$

Proof: Let $\overline{s_m}(g;x)$ be the partial sum of the (1..10) and is written as Zygmund [3],

$$|\overline{s_m}(g;x)-\overline{g}(x)|=\frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

The $\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}$ transform of $\overline{s_m}(g;x)$ is given by

$$|t_m^{\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}}(g;x)-\overline{g}|=\frac{1}{2\pi.R_m}\left|\sum_{h=0}^{m}p_{m-h}q_h\right.$$
$$\frac{1}{A_h^{\alpha+\eta}}\sum_{i=0}^{h}A_{h-i}^{\alpha-1}\ A_i^{\eta}\frac{1}{(1+\theta)^i}$$
$$\left.\times\int_0^{\pi}\psi_x(t)\left\{\sum_{j=0}^{i}\binom{i}{j}\theta^{i-j}\frac{\cos\left(j+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right|$$
$$=\int_0^{\pi}|\psi_x(t)|.|\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)|dt. \tag{4..5}$$

By using the assumptions

$$\int_0^{\pi}|\psi_x(t)||\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)|dt=O\left((1+m)^{\gamma+\frac{1}{p}}\xi\left((1+m)^{-1}\right)\right).$$

Now,

$$|t_m^{\left(\overline{NCE}\right)^{p,q;\alpha,\eta;\theta}}(g;x)-\overline{g}|=\int_0^{\pi}|\psi_x(t)||\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)|dt$$

$$= \left[\int_0^{\frac{\pi}{(1+m)}} |\psi_x(t)| + \int_{\frac{\pi}{(1+m)}}^{\pi} |\psi_x(t)|\right] |\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| dt$$

$$= |I_{1.1}| + |I_{1.2}| \, (say). \tag{4..6}$$

Using Hölder inequality, Lemma 1, condition (4..2) and $(\sin t/2)^{-1} \leq \frac{\pi}{t}$, for $0 < t \leq \pi$,

$$\begin{aligned}
|I_{1.1}| &\leq \int_0^{\frac{\pi}{(1+m)}} |\psi_x(t)| . |\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| dt \\
&\leq \left(\int_0^{\frac{\pi}{(1+m)}} \left(\frac{|\psi_x(t)|}{\xi(t)} \sin^\gamma\left(\frac{t}{2}\right)\right)^p dt\right)^{\frac{1}{p}} \\
&\times \left[\int_0^{\frac{\pi}{(1+m)}} \left(\frac{\xi(t)\, |\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)|}{\sin^\gamma(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\
&= O(1) \operatorname*{ess\,sup}_{0<t\leq \frac{\pi}{(1+m)}} [\xi(t)^q]^{\frac{1}{q}} \left[\int_0^{\frac{\pi}{(1+m)}} \left(t^{-1-\gamma}\right)^q dt\right]^{\frac{1}{q}} \\
&= O\left(\xi\left(\frac{\pi}{1+m}\right)\right) \operatorname*{ess\,sup}_{0<t\leq \frac{\pi}{(1+m)}} \left[\int_0^{\frac{\pi}{(1+m)}} \left(t^{-1-\gamma}\right)^q dt\right]^{\frac{1}{q}} \\
&= O\left(\xi\left((1+m)^{-1}\right)\right) \left[\lim_{\in\to 0} \int_{\in}^{\frac{\pi}{(1+m)}} \left(t^{-1-\gamma}\right)^q dt\right]^{\frac{1}{q}} \\
&= O\left[\xi\left((1+m)^{-1}\right)(1+m)^{\gamma+1-\frac{1}{q}}\right] \\
&= O\left[(1+m)^{\gamma+\frac{1}{p}} \xi\left((1+m)^{-1}\right)\right] \\
&\left\{\because \frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq \infty.\right\}
\end{aligned} \tag{4..7}$$

Now, in view of Lemma 2, condition (4..3), $(\sin t/2)^{-1} \leq \frac{\pi}{t}$, for $0 < t \leq \pi$, $|\sin t/2| \leq 1$ and Hölder's inequality

$$\begin{aligned}
|I_{1.2}| &\leq \int_{\frac{\pi}{(1+m)}}^{\pi} |\psi_x(t)| |\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)| dt \\
&\leq \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{\sin^\gamma(t/2)\, |\psi_x(t)|}{\xi(t) . t^\delta}\right)^p dt\right]^{\frac{1}{p}} \\
&\times \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{\xi(t) . t^\delta |\widetilde{D}_m^{p,q;\alpha,\eta;\theta}(t)|}{\sin^\gamma(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\
&\leq \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{|\psi_x(t)|}{\xi(t) . t^\delta}\right)^p dt\right]^{\frac{1}{p}} \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\frac{\xi(t) . t^{\delta-1}}{\sin^\gamma(t/2)}\right)^q dt\right]^{\frac{1}{q}} \\
&= O\left(\frac{1}{(1+m)^{-\delta}}\right) \left[\int_{\frac{\pi}{(1+m)}}^{\pi} \left(\xi(t) . t^{\delta-1-\gamma}\right)^q dt\right]^{\frac{1}{q}} \\
&= O\left(\frac{1}{(1+m)^{-\delta}}\right) \left[\int_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}} \left(\xi\left(\frac{1}{z}\right) . z^{-\delta+1+\gamma}\right)^q \frac{dz}{z^2}\right]^{\frac{1}{q}} \\
&\left\{\text{Putting } t = \frac{1}{z}\right\} \\
&= O\left(\frac{1}{(1+m)^{-\delta}} \xi\left(\frac{\pi}{1+m}\right)\right) \left[\int_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}} z^{(-\delta+1+\gamma)q-2} dz\right]^{\frac{1}{q}} \\
&= O\left(\frac{1}{(1+m)^{-\delta}} \xi\left((1+m)^{-1}\right)\right) \left[\left(\frac{z^{(-\delta+1+\gamma)q-1}}{(-\delta+1+\gamma)q-1}\right)_{\frac{1}{\pi}}^{\frac{(1+m)}{\pi}}\right]^{\frac{1}{q}} \\
&= O\left(\frac{1}{(1+m)^{-\delta}} \xi\left((1+m)^{-1}\right)\right) \left[(1+m)^{(-\delta+1+\gamma)-\frac{1}{q}}\right] \\
&= O\left(\xi\left((1+m)^{-1}\right)\right) \left[(1+m)^{\gamma+1-\frac{1}{q}}\right] \\
&= O\left((1+m)^{\gamma+\frac{1}{p}} \xi\left((1+m)^{-1}\right)\right). \\
&\left\{\because \frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq \infty.\right\}
\end{aligned} \tag{4..8}$$

Putting equations (4..7), (4..8) in (4..6), we get the proof of theorem

$$|t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}}(g;x) - \overline{g}| = O\left((1+m)^{\gamma+\frac{1}{p}} \xi\left((1+m)^{-1}\right)\right).$$

Thus

$$\begin{aligned}
\|t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}}(g;x) - \overline{g}\|_p &= \left(\int_0^{2\pi} |t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}} - \overline{g}|^p dx\right)^{\frac{1}{p}} \\
&= O\left(\int_0^{2\pi} \left((1+m)^{\gamma+\frac{1}{p}} \right.\right. \\
&\qquad \xi\left((1+m)^{-1}\right)^p dx^{\frac{1}{p}} \\
&= O\left((1+m)^{\gamma+\frac{1}{p}} \xi\left((1+m)^{-1}\right)\right).
\end{aligned}$$

which completes the proof of theorem.

5. Graphical Analysis

Example 5..0.1. *In first example, we consider a convergent oscillatory sequence*

$$a_m = (-1)^{m+1}/(m^2+m)$$

Here, the sum of above oscillatory sequence upto first hundred terms i.e.,

$$f = \sum a_m = \left\{1 - 1/6 + 1/12 - 1/20 + 1/30 - \ldots\ldots\ldots\ldots (-1)^{m+1}/(m^2+m)\right\},$$

is 0.386228. The behaviour of g, $\{s_m\}$ and Error term is observed. Here,

$$\{s_m\} = a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_m,$$

The $(N, p_m, q_m)(C, \alpha, \eta)(E, \theta)$ transform $t_m^{N^{p,q}C^{\alpha,\eta}E^\theta}$ of the m^{th} partial sums s_m of $\sum a_m$ is defined as

$$\begin{aligned}
t_m &= t_m^{N^{p,q}C^{\alpha,\eta}E^\theta} = (NCE)_m^{p,q;\alpha,\eta;\theta} \\
&= \frac{1}{R_m} \sum_{h=0}^{m} p_{m-h} q_h \frac{1}{A_h^{\alpha+\eta}} \\
&\quad \times \sum_{i=0}^{h} A_{h-i}^{\alpha-1} A_i^\eta E_i^\theta \to s \quad as\, m \to \infty \\
&= \frac{1}{R_m} \sum_{h=0}^{m} p_{m-h} q_h \frac{1}{A_h^{\alpha+\eta}} \sum_{i=0}^{h} A_{h-i}^{\alpha-1} A_i^\eta \frac{1}{(1+\theta)^i} \\
&\quad \times \sum_{j=0}^{i} \binom{i}{j} \theta^{i-j} \to s \quad as\, m \to \infty,
\end{aligned}$$

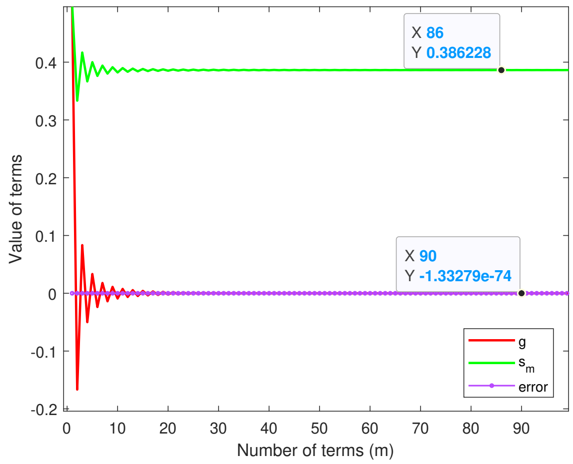

Figure 1

$$error = t_m - g$$

and variation is demonstrated by above figure. The alternating signs result in a behavior that doesn't grow indefinitely. Instead, the positive and negative terms tend to balance each other out, leading to convergence. The computed value $f \approx 0.386228$ indicates that the series converges to a specific limit.

Example 5..0.2. *Here, we consider a convergent non-oscillatory sequence*

$$a_m = 1/(n^3 + 1)$$

,

the sum of above non-oscillatory sequence upto first hundred terms i.e.,

$$f = \sum a_m = \Big\{ 1 + 1/10 + 1/28 + 1/82 + 1/244 + \ldots\ldots\ldots\ldots 1/(n^3+1) \Big\},$$

is 0.68642. The behaviour of g, $\{s_m\}$ and Error term is shown in following plot.This series converges to approximately 0.68642*. The lack of oscillation means the terms steadily contribute to the overall sum without the fluctuations as shown in Example 1.*

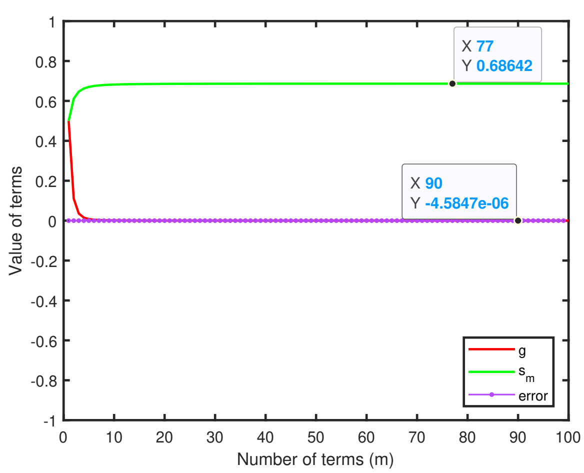

Figure 2

Similar to the first example, we track the cumulative sum s_m and analyze the error. The error behavior still tends towards zero, showing convergence.The absence of oscillation leads to a more stable convergence, as evident in the behavior of s_m.
From both of examples discussed above, we can conclude that by proposed method, error sequence in each case is converging to '0' i.e., $O(1)$ which is theoritically proved in proposed theorems. It can be observed that the oscillatory sequences can exhibit significant fluctuations in partial sums, while non-oscillatory sequences have a more predictable and steady convergence. Both types of sequences, however, converge to their respective limits.

6. Corollaries

Corollary 6..0.1. *If $\gamma = 0$ in theorem 4..0.1, then $W^{'}(L^p, \xi(t))$, $(p \geq 1)$, $(t > 0)$ reduces to Lip $(\xi(t),\ p)$.*
The approximation of $g \in$ Lip $(\xi(t),\ p)$ is

$$\|t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}}(g;x) - \overline{g}\|_p = O\left((1+m)^{\frac{1}{p}}\, \xi\left((1+m)^{-1}\right)\right).$$

Corollary 6..0.2. *If $\gamma = 0$ and $\xi(t) = t^{\beta}$, $0 < \beta \leq 1$ in theorem 4..0.1, the approximation of $g \in$ Lip $(\beta,\ p)$, $\frac{1}{p} \leq \beta \leq 1$ is*

$$\|t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}}(g;x) - \overline{g}\|_p = O\left((1+m)^{-\beta+\frac{1}{p}}\right).$$

Corollary 6..0.3. *If $\gamma = 0$, $\xi(t) = t^{\beta}$ for $0 < \beta < 1$ and if $p \to \infty$ in corollary 6..0.2, then $g \in$ Lip$(\beta,\ p)$ reduces to Lip β.*
The approximation of $g \in$ Lip β is

$$\|t_m^{(\overline{NCE})^{p,q;\alpha,\eta;\theta}}(g;x) - \overline{g}\|_\infty = O\left((1+m)^{-\beta}\right).$$

7. Conclusion

This paper focuses on the approximation of signal belongs to $W^{'}(L^p, \xi(t))$ class by $(N, p_m, q_m)(C, \alpha, \eta)(E, \theta)$ product means of conjugate Fourier series. Under general conditions, a new theorem has been established and proven. The main theorem is generalizable and can be reduced to familiar results. the error terms approach zero, suggesting that the proposed method is effective in estimating the sums of convergent sequences. This indicates that regardless of the oscillatory nature of the series, the method can yield accurate results. The results from approximating signals in the generalized weighted Lipschitz class using the proposed method of product transform hold significant potential for impact and application across various fields. This method can improve signal reconstruction, allowing for more accurate recovery of signals from incomplete or noisy data, which is particularly valuable in audio and image processing. Furthermore, its robustness against noise can enhance filtering techniques, making them more effective in real-world scenarios where data contamination is common. The method also has implications for advanced data compression, enabling more efficient storage and transmission of signals while preserving essential features and minimizing loss. Additionally, the computational efficiency gained can facilitate real-time processing applications crucial for telecommunications, live video streaming, and interactive media. Future research could explore extensions to nonlinear systems, adaptive algorithms that adjust in real-time, and integrations with machine learning models to enhance predictive capabilities.

Conflict of Interest. The authors declare that they have no conflicts of interest.

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