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Power Aggregation Operators in MADM Problems Under Pythagorean Neutrosophic Numbers Based on Hamacher t-norm and t-conorm Operations

Operadores de agregación de energía en problemas MADM bajo números neutrosóficos pitagóricos basados en operaciones t-norm y t-conorm de Hamacher

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Resumen

La noción de conjuntos neutrosóficos pitagóricos (PNS) y sus números neutrosóficos pitagóricos (PNN) asociados son útiles para explorar otro conocimiento impreciso con condiciones restringidas. Los investigadores creen que las PNS son capaces de eliminar deficiencias en las teorías existentes y son fácilmente aplicables en diversos problemas inciertos. Por otro lado, los operadores de Hamacher son útiles en la toma de decisiones de atributos múltiples (MADM). Por lo tanto, los operadores agregados neutrosóficos pitagóricos de Hamacher se consideran una herramienta poderosa para modelar la incertidumbre en problemas MADM. Este documento tiene como objetivo emplear los operadores t-norm y t-conorm de Hamachar para desarrollar algunos operadores de agregación de energía basados en PNN. Los operadores agregados incluyen el operador pitagórico de aritmética de potencia neutrosófica de Hamachar (PNHPA), el operador pitagórico de potencia neutrosófica de Hamacher (PNHPG), el operador pitagórico de potencia neutrosófica de Hamacher con promedio ponderado ordenado (PNHPOWA), el operador pitagórico de potencia neutrosófica de Hamacher con orden geométrica ponderada (PNHPOWG). También estudiamos algunas características importantes de estos operadores. Luego, al utilizar estos operadores, brindamos un enfoque para resolver los problemas de toma de decisiones de atributos múltiples bajo PNN. Al final, se muestra un ejemplo numérico para verificar el enfoque propuesto y proporcionar un análisis comparativo.

***Palabras clave*:** Operadores de Hamacher; Números neutrosóficos pitagóricos; Operadores de agregados neutrosóficos pitagóricos de Hamacher; MADM

Abstract

The notion of Pythagorean neutrosophic sets (PNSs) and their associated Pythagorean neutrosophic numbers (PNNs) are useful to explore another imprecise knowledge with the restricted conditions. Researchers believe that the PNSs are capable to remove deficiencies in the existing theories and easily applicable in various uncertain problems. On the other hand, the Hamacher operators are handy in multi-attribute decision-making (MADM). So, the Pythagorean neutrosophic Hamacher aggregate operators are considered to be the powerful tool to model uncertainty in MADM problems. This paper aims to employ the Hamachar t-norm and t-conorm operators to develop some power aggregation operators based on PNNs. The aggregate operators include the Pythagorean neutrosophic Hamachar power arithmetic (PNHPA) operator, Pythagorean neutrosophic Hamacher power geometric (PNHPG) operator, Pythagorean neutrosophic Hamacher power ordered weighted average (PNHPOWA) operator, Pythagorean neutrosophic Hamacher power ordered weighted geometric (PNHPOWG) operator. We also study some important characteristics of these operators. Then, by using these operators, we give an approach to solve the multi-attribute decision-making problems under PNNs. In the end, a numerical example is shown to verify the proposed approach and furnish a comparative analysis.

***Keywords*:** Hamacher operators; Pythagorean neutrosophic numbers; Pythagorean neutrosophic Hamacher aggregate operators; MADM

1 We are living in an era of uncertainty. So, we need to learn how to handle the uncertainty that we encounter in our daily processes. Traditional mathematics is not considered a useful tool to admit uncertain or vague knowledge. It is due to the complexity involved in human cognition. To cope with such information, researchers worked hard over the years. Finally, Zadeh [1] propounded fuzzy set (FS) theory in the year 1965. By FS, we define a particular class of objects with a spectrum of membership grades. The membership grade or membership degree of each element in a fuzzy set belongs to

the interval $[0,1]$. Mathematicians and scientists have been studied the FS quite extensively and they applied it in soil science [2], industrial engineering [3], mathematical programming [4], supply chain coordination [5], production management [6], policy analysis, and information systems [7], etc. However, FS is useful to describe the measure of belongingness of an element by a membership function. But, we cannot utilize the FS to describe the incomplete information. This led to the foundation of intuitionistic fuzzy sets (IFSs), initiated by Atanassov [8]. Atanassov's notion is very much logical and meaningful in the context of real decision-making (DM). IFS can be viewed as an extension of FS by introducing the non-membership function along with the membership function where the sum of the membership degree and the non-membership degree cannot exceed 1. The IFS has been used successfully in practical applications using the multi-attribute decision-making (MADM) method [9-12].

A MADM method is a scientific approach under uncertain information where the decision-maker prioritizes an alternative based on multiple conflict attributes. In some practical applications, where the sum of the membership and the non-membership degree of an alternative influenced by multiple attributes is greater than 1, the decision-maker failed to describe it by using FS, IFS. To eradicate such an issue, Yager [13] introduced a new mathematical tool known as the Pythagorean fuzzy set (PFS). The PFS is an improved version of IFS where the sum of the squares of the membership and the non-membership degree does not exceed 1. So, the PFS is a more powerful and significant tool as it can easily accommodate the FS and IFS information with ease. The essence of PFSs in the field of MADM problems are given as follows: Tao et al. [14] developed MADM with PFSs via IFSs and ORESTE method. Lin et al. [15] introduced the multi-attribute group decision-making based on linguistic Pythagorean fuzzy interaction Bonferroni mean aggregation operators. Paul et al. [16] used the advanced Pythagorean fuzzy weighted geometric operator in MADM for real estate company selection. Garg [17] initiated the linguistic PFSs and apply them in the MADM problems. Khan et al. [18] developed the grey method for MADM under PFS information. Wan et al. [19] introduced a novel Pythagorean group decision-making method associated with evidence theory and interactive power averaging operator. Wan et al. [20] defined Pythagorean fuzzy mathematical programming for MADM with Pythagorean fuzzy truth values. Xu et al. [21] developed the Pythagorean fuzzy interaction Muirhead means and apply it for multi-attribute GDM. But, we give some instances where the non-membership information provided by the decision-makers cannot be explained using IFS and PFS. For example, suppose in a certain domain, according to the decision-maker, the membership and the non-membership degrees of an item are 0.8 and 0.7 respectively, which is absurd in the context of IFS and PFS. To understand such knowledge, a q-rung orthopair fuzzy set (q-ROFS) is introduced by Yager [22]. In q-ROFS, the sum of the q^{th} powers of the membership and the non-membership degree is limited to 1. Practically, the q-ROFS seems to be more functional than the IFS and PFS.

None of FS, IFS, PFS, and q-ROFS is capable to describe the indeterminate, inconsistent, and incomplete information. To take care of such an issue, Smarandache [23] introduced neutrosophic set (NS) theory. In NS, every alternative is characterized by a truth-membership degree, indeterminate-membership degree, and a false-membership degree. If we consider a subclass of a NS, where the truth-membership $(\mu_A(x))$ and the false-membership degrees $(\nu_A(x))$ are dependent and the independent indeterminate-membership degree $(\xi_A(x))$ such that $0 \le \mu_A(x)+\xi_A(x)+\nu_A(x) \le 2$, which is still a more powerful tool than FS, IFS, and PFS. Although, we may witness many instances where $\mu_A(x)+\xi_A(x)+\nu_A(x) > 2$, which cannot be studied under this neutrosophic subclass. But, by the combination of NS and PFS, Jansi et al. [24] introduced a new mathematical tool known as Pythagorean neutrosophic set (PNS) with dependent truth and false neutrosophic components such that $\mu^2{}_A(x)+\xi^2{}_A(x)+\nu^2{}_A(x) \le 2$. For example, if we consider $\mu_A(x)=0.7$, $\xi_A(x)=0.8$, and $\nu_A(x)=0.9$, than it can be easily shown that $\mu_A(x)+\xi_A(x)+\nu_A(x) > 2$ but $\mu^2{}_A(x)+\xi^2{}_A(x)+\nu^2{}_A(x) = 1.94 < 2$. Some recent studies based on PNS have been proposed in [25-27]. Thus, PNS is another class of NS that has its applicability under inherent restrictions. To give a brief insight into the proposed study based on Hamacher operations, see the following Fig 1.

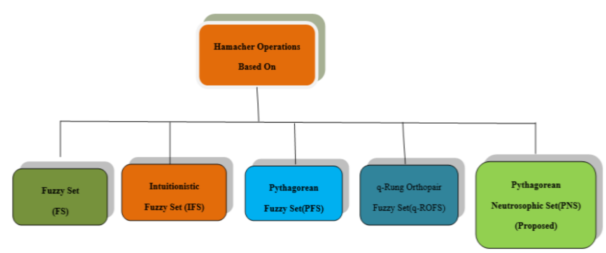

Fig 1. Hierarchy formation of the proposed study based on Hamacher operations

Hamacher [28] proposed the more sophisticated Hamacher t-norm and t-conorm operators and developed various aggregate operations that are useful in solving MADM problems. After that, Roychowdhury et al. [29] introduced some connective generators based on the Hamacher family. For group decision-making (GDM), Liu [30] proposed some Hamacher aggregation operators on interval-valued intuitionistic fuzzy numbers. Zhou et al. [31]

solved the hesitant fuzzy Hamacher aggregate operators based MADM problem. Huang [32] gives another Hamacher aggregation operator based on IFS. Liu et al. [33] propounded the Hamacher aggregation operators on neutrosophic numbers and apply them in the GDM problem. For more MADM problem-based work associated with Hamacher aggregation operators over the dual hesitant bipolar fuzzy set, dual hesitant Pythagorean fuzzy set, hesitant Pythagorean fuzzy set, Pythagorean fuzzy set, Pythagorean hesitant fuzzy set (see[34-38]) respectively. Furthermore, Wu et al. [39] initiated some Hamacher operators under single-valued neutrosophic 2-tuple linguistic information and utilize it to solve a kind of MADM problem. To evaluate land reclamation strategy for mines, Liang et al. [40] deployed Hamacher operators under linguistic neutrosophic sets. Bipolar fuzzy Hamacher aggregation operators are defined in [41]. Zhu et al. [42] presented Hamacher t-norm and t-conorm operators on hesitant fuzzy linguistic sets. Wei [43] pointed out the Hamacher power aggregation operators over the Pythagorean fuzzy set for MADM. Application of dual hesitant bipolar fuzzy set-based Hamacher aggregation operations for MADM [44]. Picture fuzzy set-based Hamacher aggregation operators used in enterprise assessment [45]. Wang et al. [46] used the notion of dual hesitant q-ROFS based Hamacher aggregation operators in scheme selection. Waseem et al. [47] utilized the m-polar fuzzy Hamacher aggregation operators in MADM. Some more Hamacher aggregation operators on q-ROFS with modified EDAS method in MADM [48]. Akram et al. [49] developed the complex picture fuzzy set-based Hamacher operators in decision-making. Ullah et al. [50] calculate the performance of search and rescue robots with an aid of T-spherical fuzzy Hamacher aggregation operators. Wang et al. [51] measured the entropy weight to assess the service quality by using the interactive Hamacher power aggregation operators under Pythagorean fuzzy information.

The main motivation to propose the present study is that there are no such previous research work that have been done so far that is based on Pythagorean neutrosophic Hamacher aggregation operators. That's why, in our study, we have introduced a new type of Hamacher aggregate operator which helps to tackle another type of information that is more often present in human cognition.

Giving preference to the decision-makers opinion to address uncertainty under different information domains, a brief analysis of different types of fuzzy sets and the PNS is exhibited in Table 1.

1.1 Motivation

In the above literature review, it has been observed that several works are using Hamacher operators under fuzzy logic and neutrosophic logic have been carried out successfully by the researchers to solve uncertain decision-making problems. Getting motivation from the research work proposed in [52], we believe that there is no such study ever been introduced that is associated with PNNs via Hamacher operators and their application in MADM problems. Keeping this in mind, we have undertaken this proposed study to encounter another type of uncertain information close to human thinking.

Table 1. Decision-makers view under the different domains of information.

Information Domain	Membership Function (m)	Non-membership Function (n)	Indeterminacy function (p)	Remark
FSs	Yes	No	No	$0 \le m \le 1$
IFSs	Yes	Yes	No	$0 \le m \le 1$ $0 \le n \le 1$ $0 \le m+n \le 1$
PFSs	Yes	Yes	No	$0 \le m^2 + n^2 \le 1$
q-ROFSs	Yes	Yes	No	$0 \le m^q + n^q \le 1, q \ge 1$
PNSs	Yes	Yes	Yes	$0 \le m^2 + n^2 + p^2 \le 2$

1.2 Framework of the Paper

The rest of the paper is arranged as follows:

In section 2, we have studied some preliminary concepts related to Pythagorean neutrosophic numbers and their properties. We also defined Hamacher operations based on Pythagorean neutrosophic numbers. Section 3 includes two types of Pythagorean neutrosophic Hamacher power aggregation operators and their important properties. A new decision-making model based on Hamacher operators via Pythagorean neutrosophic numbers is exhibited in section 4. This proposed model is successfully executed with the help of a practical example in section 5. Finally, we conclude section 6.

2. Preliminaries

2.1 Pythagorean Neutrosophic Set

In this section, we first give the basic notion of the Pythagorean neutrosophic set (PNS) [27] and its associated Pythagorean neutrosophic number (PNN). Basic operations on PNNs are newly introduced. Then, the novel score and accuracy function is based on PNNs. Afterward, a new comparison method between two PNNs is developed.

Definition 2.1.1[24, 27]

Let X be a set of the universe. A Pythagorean neutrosophic set A on X is an object of the form

$A=\left\{\left\langle x, \mu_A(x), \xi_A(x), \nu_A(x)\right\rangle : x \in X\right\}$, where $\mu_A(x)$, $\xi_A(x)$ and $\nu_A(x)$ respectively denote the acceptance degree, indeterminate degree and the non-acceptance degree such that $\mu_A(x), \xi_A(x), \nu_A(x) \in [0,1]$, $0 \le \mu_A(x) + \nu_A(x) \le 1$ and $0 \le \mu^2{}_A(x) + \xi^2{}_A(x) + \nu^2{}_A(x) \le 2$, for all $x \in X$.

It is to be noted that, $\langle \mu, \xi, \nu \rangle$ denotes a Pythagorean neutrosophic number corresponding to a Pythagorean neutrosophic set.

Definition 2.1.2 [24, 27]

Let $A=\left\{\left\langle x,\mu_A(x),\xi_A(x),\nu_A(x)\right\rangle : x\in X\right\}$ and $B=\left\{\left\langle x,\mu_B(x),\xi_B(x),\nu_B(x)\right\rangle : x\in X\right\}$ be two Pythagorean neutrosophic sets over X . Then, we consider the following properties:

(i) $A\bigcup B=\left\{\left\langle x,\max\left(\mu_A(x),\mu_B(x)\right),\max\left(\xi_A(x),\xi_B(x)\right),\min\left(\nu_A(x),\nu_B(x)\right)\right\rangle : x\in X\right\}$

(ii) $A\bigcap B=\left\{\left\langle x,\min\left(\mu_A(x),\mu_B(x)\right),\min\left(\xi_A(x),\xi_B(x)\right),\max\left(\nu_A(x),\nu_B(x)\right)\right\rangle : x\in X\right\}$

(iii) $A^c=\left\{\left\langle x,\nu_A(x),\xi_A(x),\mu_A(x)\right\rangle : x\in X\right\}$

(iv) $A\subseteq B$ if and only if $\mu_A(x)\le\mu_B(x),\ \xi_A(x)\le\xi_B(x)$ and $\nu_A(x)\ge\nu_B(x)$

Definition 2.1.3 Let $\partial_1=\langle\mu_1,\xi_1,\nu_1\rangle$, $\partial_2=\langle\mu_2,\xi_2,\nu_2\rangle$ and $\partial=\langle\mu,\xi,\nu\rangle$ be three PNNs. Then, some basic operations on them are defined as follows:

(i) $\partial_1\oplus\partial_2=\left\langle\sqrt{\mu_1^2+\mu_2^2-\mu_1^2.\mu_2^2},\ \sqrt{\xi_1^2+\xi_2^2-\xi_1^2.\xi_2^2},\ \nu_1.\nu_2\right\rangle$

(ii) $\partial_1\otimes\partial_2=\left\langle\mu_1.\mu_2,\ \xi_1.\xi_2,\ \sqrt{\nu_1^2+\nu_2^2-\nu_1^2.\nu_2^2}\right\rangle$

(iii) $\eta\partial=\left\langle\sqrt{1-\left(1-\mu^2\right)^\eta},\ \sqrt{1-\left(1-\xi^2\right)^\eta},\ \nu^\eta\right\rangle,\eta>0$

(iv) $\partial^\eta=\left\langle\mu^\eta,\ \xi^\eta\sqrt{1-\left(1-\nu^2\right)^\eta}\right\rangle,\eta>0$

(V) $\partial^c=\langle\nu,\xi,\mu\rangle$

Definition 2.1.4

Let $\partial=\langle\mu,\xi,\nu\rangle$ be a PNN. Then, a score function $\$$ on ∂ is represented as follows:

$$\$(\partial)=\frac{1}{3}\left(1+\mu^2+\xi^2-\nu^2\right),\ \$(\partial)\in[0,1]$$

Definition 2.1.5

Let $\partial=\langle\mu,\xi,\nu\rangle$ be a PNN. Then, accuracy function # on ∂ is represented as follows:

$$\#(\partial)=\frac{1}{2}\left(\mu^2+\xi^2+\nu^2\right),\ \#(\partial)\in[0,1]$$

Definition 2.1.6

Let $\partial_1=\langle\mu_1,\xi_1,\nu_1\rangle$, $\partial_2=\langle\mu_2,\xi_2,\nu_2\rangle$ be two PNNs. Their scores and accuracy functions are represented by $\$(\partial_1),\$(\partial_2)$ and $\#(\partial_1),\#(\partial_2)$ respectively. If $\$(\partial_1)<\$(\partial_2)$, then ∂_1 is smaller than ∂_2 i.e. $\partial_1<\partial_2$; if $\$(\partial_1)=\$(\partial_2)$, then we have the following:

(i) If $\#(\partial_1)=\#(\partial_2)$, then $\partial_1=\partial_2$

(ii) If $\#(\partial_1)<\#(\partial_2)$, then $\partial_1<\partial_2$

2.2 Hamacher Operations on Pythagorean Neutrosophic Numbers

Definition 2.2.1 [28] Let α,β be any two real numbers where $(\alpha,\beta)\in(0,1)\times(0,1)$ with $\rho>0$. Then Hamacher's t-norms and t-conorms are defined as

$$H(\alpha,\beta)=\alpha\otimes\beta=\frac{\alpha\beta}{\rho+(1-\rho)(\alpha+\beta-\alpha\beta)}$$

and $$H^*(\alpha,\beta)=\alpha\oplus\beta=\frac{\alpha+\beta-\alpha\beta-(1-\rho)\alpha\beta}{1-(1-\rho)\alpha\beta}$$

respectively.

Definition 2.2.2

Let $\partial_1=\langle\mu_1,\xi_1,\nu_1\rangle$, $\partial_2=\langle\mu_2,\xi_2,\nu_2\rangle$ and $\partial=\langle\mu,\xi,\nu\rangle$ be three PNNs and $\rho>0$. Based on definition **2.2.1**, some basic operations on PNNs are defined as follows:

$$\partial_1\oplus_H\partial_2=\left(\left\langle\sqrt{\frac{\mu_1^2+\mu_2^2-\mu_1^2.\mu_2^2-(1-\rho)\mu_1^2.\mu_2^2}{1-(1-\rho)\mu_1^2.\mu_2^2}},\sqrt{\frac{\xi_1^2+\xi_2^2-\xi_1^2.\xi_2^2-(1-\rho)\xi_1^2.\xi_2^2}{1-(1-\rho)\xi_1^2.\xi_2^2}},\frac{\nu_1.\nu_2}{\rho+(1-\rho)\left(\nu_1^2+\nu_2^2-\nu_1^2.\nu_2^2\right)}\right\rangle\right)\text{......(1)}$$

$$\partial_1\otimes_H\partial_2=\left(\left\langle\frac{\mu_1.\mu_2}{\rho+(1-\rho)\left(\mu_1^2+\mu_2^2-\mu_1^2.\mu_2^2\right)},\frac{\xi_1.\xi_2}{\rho+(1-\rho)\left(\xi_1^2+\xi_2^2-\xi_1^2.\xi_2^2\right)},\sqrt{\frac{\nu_1^2+\nu_2^2-\nu_1^2.\nu_2^2-(1-\rho)\nu_1^2.\nu_2^2}{1-(1-\rho)\nu_1^2.\nu_2^2}},\right\rangle\right)\text{.......(2)}$$

$$\kappa\partial=\left(\left\langle\sqrt{\frac{\left(1+(\rho-1)\mu^2\right)^\kappa-\left(1-\mu^2\right)^\kappa}{\left(1+(\rho-1)\mu^2\right)^\kappa-\left((\rho-1)\left(1-\mu^2\right)\right)^\kappa}},\sqrt{\frac{\left(1+(\rho-1)\xi^2\right)^\kappa-\left(1-\xi^2\right)^\kappa}{\left(1+(\rho-1)\xi^2\right)^\kappa-\left((\rho-1)\left(1-\xi^2\right)\right)^\kappa}},\frac{\sqrt{\rho}\,\nu^\kappa}{\sqrt{1+(\rho-1)\left(1-\nu^2\right)^\kappa-(\rho-1)\nu^{2\kappa}}}\right\rangle\right)\text{........(3)}$$

$$\partial^\kappa=\left(\left\langle\frac{\sqrt{\rho}\,\mu^\kappa}{\sqrt{1+(\rho-1)\left(1-\mu^2\right)^\kappa-(\rho-1)\mu^{2\kappa}}},\frac{\sqrt{\rho}\,\xi^\kappa}{\sqrt{1+(\rho-1)\left(1-\xi^2\right)^\kappa-(\rho-1)\xi^{2\kappa}}},\sqrt{\frac{\left(1+(\rho-1)\nu^2\right)^\kappa-\left(1-\nu^2\right)^\kappa}{\left(1+(\rho-1)\nu^2\right)^\kappa-\left((\rho-1)\left(1-\nu^2\right)\right)^\kappa}}\right\rangle\right)\text{........(4)}$$

3. Pythagorean Neutrosophic Hamacher Power Aggregation Operators

3.1. Pythagorean Neutrosophic Hamacher Power Arithmetic Aggregation Operators and their Properties

Definition 3.1.1 Let $\partial_l=\langle\mu_l,\xi_l,\nu_l\rangle\ (l=1,2,...,n)$ denotes a collection of PNNs. Then, we define the Pythagorean neutrosophic Hamacher power arithmetic (PNHPA) operator as follows:

$$PNHPA(\partial_1,\partial_2,\partial_3,......,\partial_n)=\oplus_{l=1}^n\left(\partial_l\right)^{\frac{\left(\left(1+T(\partial_l)\right)\right)}{\sum_{l=1}^n\left(1+T(\partial_l)\right)}}$$

, where $l=1,2,...,n$, $T(\partial_l)=\sum_{m=1,\,m\ne l}^n\sup(\partial_l,\partial_m)$ with the properties:

(i) $\sup(\partial_l,\partial_m)\in[0,2]$ where $l\ne m$

(ii) $\sup(\partial_l,\partial_m)=\sup(\partial_m,\partial_l)$

(iii) $\sup\left(\partial_l,\partial_m\right)\geq\sup\left(\partial_p,\partial_q\right)$ where $l\neq m\neq p\neq q$, if $d\left(\partial_l,\partial_m\right)<d\left(\partial_p,\partial_q\right)$, where d is the distance measure.

(iv) $\sup(\partial_l,\partial_m)<\sup(\partial_l,\partial_p)+\sup(\partial_p,\partial_m)$

Example 3.1.2 To make proper justification of the properties mentioned in the definition **3.1.1**, we consider the following example:

Let $\partial_1=\left(0.6,0.5,0.3\right)$, $\partial_2=\left(0.7,0.8,0.2\right)$, $\partial_3=\left(0.5,0.9,0.4\right)$, and $\partial_4=\left(0.6,0.8,0.3\right)$ be 4 PNNs. Then

$$\sup(\partial_1,\partial_2)=2-d\left(\partial_1,\partial_2\right)=2-\frac{\left|(0.6)^2-(0.7)^2\right|+\left|(0.5)^2-(0.8)^2\right|+\left|(0.3)^2-(0.2)^2\right|}{2}$$
$$=1.715$$

Similarly, $\sup(\partial_1,\partial_3)=1.63$, $\sup(\partial_2,\partial_3)=1.735$, and $\sup(\partial_3,\partial_4)=1.825$

Therefore, (i) and (ii) hold true.

Now, $d\left(\partial_3,\partial_4\right)$=0.175 and $d\left(\partial_1,\partial_2\right)=0.285$

Clearly, $d\left(\partial_3,\partial_4\right)<d\left(\partial_1,\partial_2\right)$ but $\sup(\partial_3,\partial_4)>\sup(\partial_1,\partial_2)$ which gives (iii) true.

Also, $\sup(\partial_1,\partial_2)<\sup(\partial_1,\partial_3)+\sup(\partial_3,\partial_2)$ justify (iv)

Based on the definition **3.1.1,** we discuss the following:

Theorem 3.1.3 The aggregate value using PNHPA operator is also a PNN, where

$$PNHPA(\partial_1,\partial_2,\partial_3,......,\partial_n)=\oplus_{l=1}^{n}\left(\partial_l\right)^{\frac{\left(\left(1+T\left(\partial_l\right)\right)\right)}{\sum_{l=1}^{n}\left(1+T\left(\partial_l\right)\right)}}$$
$$=\left\langle\begin{array}{c}\sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}-\prod_{l=1}^{n}\left(1-\mu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+(\rho-1)+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(1-\mu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}},\\ \sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}-\prod_{l=1}^{n}\left(1-\xi_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+(\rho-1)+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(1-\xi_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}},\\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\nu_l\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\nu_l^2\right)\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(\nu_l\right)^{\frac{3(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}}\end{array}\right\rangle \quad ......(5)$$

where $T\left(\partial_l\right)=\sum_{l=1,\,m\neq l}^{n}\sup\left(\partial_l,\partial_m\right)$.

Definition 3.1.4 Let $\partial_l=\left\langle\mu_l,\xi_l,\nu_l\right\rangle\left(l=1,2,...,n\right)$ denotes a collection of PNNs, $\varpi=\left(\varpi_1,\varpi_2,....,\varpi_n\right)^t$ be the weight vector of ∂_l, and $\varpi_l>0$ such that $\sum_{l=1}^{n}\varpi_l=1$. Then, the Pythagorean neutrosophic Hamacher power weighted arithmetic (PNHPWA) operator is a mapping $F^n\rightarrow F$ such that

$$PNHPWA\left(\partial_1,\partial_2,....,\partial_n\right)=\oplus_{l=1}^{n}\varpi_l\left(\partial_l\right)^{\frac{\varpi_l\left(\left(1+T\left(\partial_l\right)\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_l\right)\right)}}$$
$$\left\langle\begin{array}{c}\sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}-\prod_{l=1}^{n}\left(1-\mu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+(\rho-1)+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(1-\mu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}},\\ \sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}-\prod_{l=1}^{n}\left(1-\xi_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+(\rho-1)+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(1-\xi_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}},\\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\nu_l\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\nu_l^2\right)\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}+(\rho-1)\prod_{l=1}^{n}\left(\nu_l\right)^{\frac{3\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}}\end{array}\right\rangle \quad ......(6)$$

where, $T\left(\partial_l\right)=\sum_{l=1,\,m\neq l}^{n}\varpi_l\sup\left(\partial_l,\partial_m\right)$.

If $\varpi=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)^t$ then the PNHPWA operator reduces to PNHPA operator where $T\left(\partial_l\right)=\frac{1}{n}\sum_{l=1,\,m\neq l}^{n}\sup\left(\partial_l,\partial_m\right)$.

Now, we consider the following propositions, based on PNHPWA operator that can be easily proved.

Proposition 3.1.5 (Idempotency) If $\partial_l=\partial\left(l=1,2,..,n\right)$ i.e. all PNNs are equal, then $PNHPWA\left(\partial_1,\partial_2,....,\partial_n\right)=\partial$.

Proposition 3.1.6 (Boundedness) Let $\partial_l=\left\langle\mu_l,\xi_l,\nu_l\right\rangle\left(l=1,2,...,n\right)$ be a collection of PNNs, where $\min_{l=1}^{n}\left(\partial_l\right)=\partial^-$, $\max_{l=1}^{n}\left(\partial_l\right)=\partial^+$ then $\partial^-\leq PNHPWA\left(\partial_1,\partial_2,....,\partial_n\right)\leq\partial^+$.

Proposition 3.1.7 (Monotonicity) Let ∂_l and ∂'_l $\left(l=1,2,...,n\right)$ be two set of PNNs having the same dimension. If $\partial_l\leq\partial'_l$ for all l, then $PNHPWA\left(\partial_1,\partial_2,....,\partial_n\right)\leq PNHPWA\left(\partial'_1,\partial'_2,....,\partial'_n\right)$.

Definition 3.1.8 Let $\partial_l=\left\langle\mu_l,\xi_l,\nu_l\right\rangle\left(l=1,2,...,n\right)$ be a collection of PNNs. Then, the Pythagorean neutrosophic

Hamacher power ordered weighted arithmetic(PNHPOWA) operator of dimension n is a mapping $PNHPOWA: F^n \to F$ with weight vector $\varpi = \left(\varpi_1, \varpi_2, ..., \varpi_n\right)^t$ such that $\varpi_l > 0$ such that $\sum_{l=1}^{n} \varpi_l = 1.$

Then,

$$PNHPOWA\left(\partial_1, \partial_2, \partial_3, ..., \partial_n\right) = \oplus_{l=1}^{n} \varpi_l \left(\partial_{\sigma(l)}\right)^{\frac{\varpi_l\left(\left(1+T\left(\partial_{\sigma(l)}\right)\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}$$

$$= \left(\left\langle \begin{array}{c} \sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}} - \prod_{l=1}^{n}\left(1-\mu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\mu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+(\rho-1)+T\left(\partial_{\sigma(l)}\right)\right)}} + (\rho-1)\prod_{l=1}^{n}\left(1-\mu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}}, \\ \sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}} - \prod_{l=1}^{n}\left(1-\xi_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\xi_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+(\rho-1)+T\left(\partial_{\sigma(l)}\right)\right)}} + (\rho-1)\prod_{l=1}^{n}\left(1-\xi_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}}, \\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\nu_{\sigma(l)}\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\nu_{\sigma(l)}^2\right)\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}} + (\rho-1)\prod_{l=1}^{n}\left(\nu_{\sigma(l)}\right)^{\frac{3\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}} \end{array} \right\rangle \right)$$

Where $\sigma(l)$ is a permutation of $l = 1, 2, ..., n$ such that $\partial_{\sigma(l-1)} \geq \partial_{\sigma(l)}$ for all l, and $\varpi_l \left(l = 1, 2, ..., n\right)$ is a collection of weights.

3.2. Pythagorean Neutrosophic Hamacher Power Geometric Aggregation Operators and their Properties

Definition 3.2.1 Let $\partial_l = \left\langle \mu_l, \xi_l, \nu_l \right\rangle \left(l = 1, 2, ..., n\right)$ denotes a collection of PNNs. Then, we define the Pythagorean neutrosophic Hamacher power geometric (PNHPG) operator as follows:

$$PNHPG(\partial_1, \partial_2, \partial_3, ..., \partial_n) = \otimes_{l=1}^{n} \left(\partial_l\right)^{\frac{\left(\left(1+T\left(\partial_l\right)\right)\right)}{\sum_{l=1}^{n}\left(1+T\left(\partial_l\right)\right)}},$$

where $l = 1, 2, ..., n$, $T\left(\partial_l\right) = \sum_{m=1,\, m\neq l} \sup\left(\partial_l, \partial_m\right)$ with the properties:

(i) $\sup\left(\partial_l, \partial_m\right) \in [0, 2]$

(ii) $\sup\left(\partial_l, \partial_m\right) = \sup\left(\partial_m, \partial_l\right)$

(iii) $\sup\left(\partial_l, \partial_m\right) \geq \sup\left(\partial_p, \partial_q\right)$, if $d\left(\partial_l, \partial_m\right) < d\left(\partial_p, \partial_q\right)$, where d is the distance measure.

Based on definition **3.2.1**, we discuss the following:

Theorem 3.2.2 The aggregate value using PNHPG operator is also a PNN, where

$$PNHPG(\partial_1, \partial_2, \partial_3, ..., \partial_n) = \otimes_{l=1}^{n} \left(\partial_l\right)^{\frac{\left(\left(1+T\left(\partial_l\right)\right)\right)}{\sum_{l=1}^{n}\left(1+T\left(\partial_l\right)\right)}}$$

$$= \left(\left\langle \begin{array}{c} \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\mu_l\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\mu_l^2\right)\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(\mu_l\right)^{\frac{3(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}}, \\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\xi_l\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\xi_l^2\right)\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(\xi_l\right)^{\frac{3(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}}, \\ \sqrt{\dfrac{\prod_{l=1}^{n}\left(1+(\rho-1)\nu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}} - \prod_{l=1}^{n}\left(1-\nu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}{\prod_{l=1}^{n}\left(1+(\rho-1)\nu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+(\rho-1)+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(1-\nu_l^2\right)^{\frac{(1+T(\partial_l))}{\sum_{l=1}^{n}(1+T(\partial_l))}}}} \end{array} \right\rangle \right) \text{....(8)}$$

where $T\left(\partial_l\right) = \sum_{m=1,\, m\neq l}^{n} \sup\left(\partial_l, \partial_m\right)$.

Definition 3.2.3 Let $\partial_l = \left\langle \mu_l, \xi_l, \nu_l \right\rangle \left(l = 1, 2, ..., n\right)$ denotes a collection of PNNs, $\varpi = \left(\varpi_1, \varpi_2, ..., \varpi_n\right)^t$ be the weight vector of ∂_l, and $\varpi_l > 0$ such that $\sum_{l=1}^{n} \varpi_l = 1$. Then, the Pythagorean neutrosophic Hamacher power weighted geometric (PNHPWG) operator is a mapping $F^n \to F$ such that

$$PNHPWG\left(\partial_1, \partial_2, ..., \partial_n\right) = \otimes_{l=1}^{n} \varpi_l \left(\partial_l\right)^{\frac{\varpi_l\left(\left(1+T\left(\partial_l\right)\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_l\right)\right)}}$$

$$= \left(\left\langle \begin{array}{c} \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\mu_l\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\mu_l^2\right)\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(\mu_l\right)^{\frac{3\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}}, \\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\xi_l\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\sqrt{\prod_{l=1}^{n}\left(1+(\rho-1)\left(1-\xi_l^2\right)\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(\xi_l\right)^{\frac{3\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}}, \\ \sqrt{\dfrac{\prod_{l=}^{n}\left(1+(\rho-1)\nu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}} - \prod_{l=1}^{n}\left(1-\nu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}{\prod_{l=}^{n}\left(1+(\rho-1)\nu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+(\rho-1)+T(\partial_l))}} + (\rho-1)\prod_{l=1}^{n}\left(1-\nu_l^2\right)^{\frac{\varpi_l(1+T(\partial_l))}{\sum_{l=1}^{n}\varpi_l(1+T(\partial_l))}}}} \end{array} \right\rangle \right) \text{......(9)}$$

where, $T\left(\partial_l\right) = \sum_{m=1,\, m\neq l}^{n} \varpi_l \sup\left(\partial_l, \partial_m\right)$.

Now, we consider the following propositions, based on PNHPWA operator that can be easily proved.

Proposition 3.2.4. (Idempotency) If $\partial_l = \partial\left(l = 1, 2, .., n\right)$ i.e. all PNNs are equal, then $PNHPWG\left(\partial_1, \partial_2, ..., \partial_n\right) = \otimes_{l=1}^{n} \partial_l = \partial$.

Proposition 3.2.5. (Boundedness) Let $\partial_l = \left\langle \mu_l, \xi_l, \nu_l \right\rangle \left(l = 1, 2, ..., n\right)$ be a collection of PNNs, where $\min_{l=1}^{n}\left(\partial_l\right)=\partial^-$, $\max_{l=1}^{n}\left(\partial_l\right)=\partial^+$ then $\partial^- \le PNHPWG\left(\partial_1,\partial_2,....,\partial_n\right)\le \partial^+$.

Proposition 3.2.6. (Monotonicity) Let ∂_l and ∂'_l $\left(l=1,2,...,n\right)$ be two set of PNNs having the same dimension. If $\partial_l \le \partial'_l$ for all l, then $PNHPWG\left(\partial_1,\partial_2,....,\partial_n\right)\le PNHPWG\left(\partial'_1,\partial'_2,....,\partial'_n\right)$.

Definition 3.2.7 Let $\partial_l=\left\langle \mu_l,\xi_l,\nu_l\right\rangle\left(l=1,2,...,n\right)$ be a collection of PNNs. Then, the Pythagorean neutrosophic Hamacher power ordered weighted geometric(PNHPOWG) operator of dimension n is a mapping $PNHPOWG: F^n \to F$ with weight vector $\varpi=\left(\varpi_1,\varpi_2,....,\varpi_n\right)^t$ such that $\varpi_l>0$ such that $\sum_{l=1}^{n}\varpi_l=1.$

Then,

$$PNHPOWG\left(\partial_1,\partial_2,\partial_3,...,\partial_n\right)=\otimes_{l=1}^{n}\varpi_l\left(\partial_{\sigma(l)}\right)^{\frac{\varpi_l\left(\left(1+T\left(\partial_{\sigma(l)}\right)\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}$$

$$=\left(\left\langle \begin{array}{c} \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\mu_{\sigma(l)}\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\sqrt{\prod_{l=1}^{n}\left(1+\left(\rho-1\right)\left(1-\mu_{\sigma(l)}^2\right)\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}+\left(\rho-1\right)\prod_{l=1}^{n}\left(\mu_{\sigma(l)}\right)^{\frac{3\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}}, \\ \dfrac{\sqrt{\rho}\prod_{l=1}^{n}\left(\xi_{\sigma(l)}\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\sqrt{\prod_{l=1}^{n}\left(1+\left(\rho-1\right)\left(1-\xi_{\sigma(l)}^2\right)\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}+\left(\rho-1\right)\prod_{l=1}^{n}\left(\xi_{\sigma(l)}\right)^{\frac{3\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}}, \\ \sqrt{\dfrac{\prod_{l=}^{n}\left(1+\left(\rho-1\right)\nu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}-\prod_{l=1}^{n}\left(1-\nu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}{\prod_{l=}^{n}\left(1+\left(\rho-1\right)\nu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+\left(\rho-1\right)+T\left(\partial_{\sigma(l)}\right)\right)}}+\left(\rho-1\right)\prod_{l=1}^{n}\left(1-\nu_{\sigma(l)}^2\right)^{\frac{\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}{\sum_{l=1}^{n}\varpi_l\left(1+T\left(\partial_{\sigma(l)}\right)\right)}}}} \end{array}\right\rangle\right) \quad .....(10)$$

Where $\sigma\left(l\right)$ is a permutation of $l=1,2,...,n$ such that $\partial_{\sigma(l-1)}\ge\partial_{\sigma(l)}$ for all l, $\varpi_l\left(l=1,2,...,n\right)$ is a collection of weights.

4. Decision-Making Model Based on Hamacher Operators under Pythagorean Neutrosophic Information

Decision-making is an important issue in many conflicting computational methods. Recently, the use of multi-attribute decision-making (in short MADM) is considered to be an effective and popular scientific tool for real decision-making under uncertainty. It provides high precision and accuracy to reach an ultimate goal while handling various complex decision-making problems. According to researchers and scientists, the MADM mechanism is a powerful, significant, effective, and systematic way to deal with problems that involve several alternatives influenced by several conflicting criteria. Decision-makers can make accurate and flexible decisions by using MADM approaches in different fields. So, in this section, we attempt to make use of the Hamacher aggregation operators based MADM technique under the Pythagorean neutrosophic environment. For this, we consider the following MADM model that helps us to make precise decisions under Pythagorean neutrosophic information:

Let us consider the set of attributes $A=\left\{A_1,A_2,......,A_p\right\}$, set of alternatives $B=\left\{B_1,B_2,......,B_q\right\}$, and $\varpi=\left\{\varpi_1,\varpi_2,......,\varpi_q\right\}$ be the weight vector of attributes such that $\varpi_k>0$, $k=1,2,...,q$, $\sum_{k=1}^{q}\varpi_k=1$. Moreover, let

$$D_M=\left(d_{ij}\right)_{q\times p}=\left(\left\langle \mu_{ij},\xi_{ij},\nu_{ij}\right\rangle\right)_{q\times p}=$$

$$\begin{pmatrix} \left\langle \mu_{11},\xi_{11},\nu_{11}\right\rangle & \left\langle \mu_{12},\xi_{12},\nu_{12}\right\rangle & .... & \left\langle \mu_{1p},\xi_{1p},\nu_{1p}\right\rangle \\ \left\langle \mu_{21},\xi_{21},\nu_{21}\right\rangle & \left\langle \mu_{22},\xi_{22},\nu_{22}\right\rangle & .... & \left\langle \mu_{2p},\xi_{2p},\nu_{2p}\right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \mu_{q1},\xi_{q1},\nu_{q1}\right\rangle & \left\langle \mu_{q2},\xi_{q2},\nu_{q2}\right\rangle & .... & \left\langle \mu_{qp},\xi_{qp},\nu_{qp}\right\rangle \end{pmatrix}_{q\times p}$$

be a Pythagorean neutrosophic decision matrix. The entries $\left\langle \mu_{ij},\xi_{ij},\nu_{ij}\right\rangle_{q\times p}$ corresponding to the i^{th} alternative that satisfies the j^{th} attribute under Pythagorean neutrosophic environment satisfy the condition $\mu_{ij},\xi_{ij},\nu_{ij}\in\left[0,1\right]$ such that $0\le\mu_{ij}+\nu_{ij}\le 1$ and $0\le\mu^2_{ij}+\xi^2_{ij}+\nu^2_{ij}\le 2,\ for\ i=1,2,..,q;\ j=1,2,..,p$.

In the following, we consider a stepwise MADM model based on **PNHPWA** (or **PNHPWG**) operator for the scientific evaluation of emerging performing insurance companies under Pythagorean neutrosophic environment.

Step 1 Input the imprecise data provided by the decision-maker under the PNS environment in the form of a set of attributes $A=\left\{A_1,A_2,......,A_p\right\}$, set of alternatives $B=\left\{B_1,B_2,......,B_q\right\}$. Then obtain the decision matrix D_M provided that the set of alternatives is influenced by the set of attributes. Also, we consider the set of weight vectors associated with the set of attributes as $\varpi=\left\{\varpi_1,\varpi_2,......,\varpi_p\right\}$, where $\sum_{i=1}^{p}\varpi_i=1.$

Step 2 Using definition **3.1.1,** calculate the following supports:

$$\sup\left(\partial_{ij},\partial_{ik}\right)=2-d\left(\partial_{ij},\partial_{ik}\right),\ where\ i=1,2,..,p;\ j,k=1,2,..,q$$

. And we calculate the distance measure $d\left(\partial_{ij},\partial_{ik}\right)$ with normalized Hamming distance under PNS as

$$d\left(\partial_{ij},\partial_{ik}\right)=\frac{\left(\left|\mu_{ij}^{2}-\mu_{ik}^{2}\right|+\left|\xi_{ij}^{2}-\xi_{ik}^{2}\right|+\left|\nu_{ij}^{2}-\nu_{ik}^{2}\right|\right)}{2},\ i=1,2,..,p;\ j,k=1,2,..,q$$

Step 3 Utilizing the weights ϖ_i associated with the attributes $A_i\,(i=1,2,..,p)$, calculate the weighted support $T\left(\partial_{ij}\right)$ defined by $T\left(\partial_{ij}\right)=\sum_{k=1,\ j\neq k}^{p}\varpi_k\sup\left(\partial_{ij},\partial_{ik}\right),\ j=1,2,..,q$.

Also, calculate the weight W_{ij} associated with the Pythagorean neutrosophic number ∂_{ij} $\left(i=1,2,..,p;\ j=1,2,..,q\right)$

$$W_{ij}=\frac{\varpi_i\left(1+T\left(\partial_{ij}\right)\right)}{\sum_{i=1}^{p}\varpi_i\left(1+T\left(\partial_{ij}\right)\right)},\ i=1,2,..,p;\ j=1,2,..,q,$$

where $W_{ij}\geq 0$, and $\sum_{i=1}^{p}W_{ij}=1$.

Step 4 Calculate the aggregate values of the decision-matrix based on the MADM problem using the **PNHPWA** (or **PNHPWG**) operator defined in **section 3**.

Step 5 Find the scores $\$\left(\partial_i\right)$ of the aggregate PNNs ∂_i. Then we arrange all the alternatives according to the magnitude of their scores and finally select the best choice having maximum scores. In case of a tie i.e. if $\$\left(\partial_i\right)=\$\left(\partial_j\right),\ i\neq j$, then we need to calculate the accuracy values $\#\left(\partial_i\right)$ and $\#\left(\partial_j\right)$. Once again rank them according to the magnitude of their accuracy values and thus choose the best option having the highest accuracy value.

Step 6 End

5. Numerical Example

Nowadays people from all over the world are interested to ensure avoiding any kind of loss that may occur due to uncertain phenomena in everyday life. To safeguard their life and reduce loss in business or any kind of profession, they searching for a good performing insurance company. Insurance provides financial support and reduces sudden risk in any profession in life. So, we always are looking for a good performing insurance company that gives maximum safeguard with a less yearly premium. But, our problem is, how to select the best insurance company with an affordable yearly premium that provides cover in any loss due to uncertainty. To run an insurance company effectively there is a lot of issues such as risk management (RM), organizational performance (OP), strategic decisions (SD), and corporate governance (CG). RM is a process of analyzing, assessing, monitoring, and controlling risk for better decision-making and it is an important factor for the survival and profitability of an insurance company. OP mitigates the goals and objectives of a company. SD is concerned with organizational activities and they are uncertain. CG provides the framework for achieving a company's goals and objectives. Utilizing the stepwise MADM model proposed in section 4, we give a practical example to select the best performing insurance company influenced by certain conflicting criteria under the PNS environment:

Step 1

We consider a set of five insurance companies(IC) or alternatives denoted by the set $C=\left\{c_1,c_2,c_3,c_4,c_5\right\}$ and a panel of experts suggests a set of four attributes or criteria denoted by the set $A=\left\{a_1,a_2,a_3,a_4\right\}$ to select the best performing insurance company. The four attributes are defined as: a_1 =RM, a_2 =OP, a_3 =SD, and a_4 =CG. And the weighting vector of the four attributes is denoted by $\varpi=\left(0.3,0.2,0.1,0.4\right)^T$. To select the best alternative which fulfills all the given criteria, we utilize the proposed MADM model given in section 4.

The evaluation of the five IC can be evaluated by the PNS information by the expert under the four attributes and it can be represented by the following decision matrix:

$$D_M=\begin{array}{c c}
 & \begin{array}{cccc} \quad a_1 \quad & \quad a_2 \quad & \quad a_3 \quad & \quad a_4 \quad \end{array} \\
\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} &
\begin{bmatrix}
\langle 0.3,0.4,0.5\rangle & \langle 0.4,0.5,0.2\rangle & \langle 0.1,0.5,0.3\rangle & \langle 0.2,0.5,0.6\rangle \\
\langle 0.3,0.6,0.5\rangle & \langle 0.6,0.2,0.4\rangle & \langle 0.4,0.6,0.3\rangle & \langle 0.4,0.5,0.3\rangle \\
\langle 0.5,0.7,0.3\rangle & \langle 0.4,0.3,0.5\rangle & \langle 0.3,0.1,0.5\rangle & \langle 0.6,0.3,0.2\rangle \\
\langle 0.4,0.3,0.6\rangle & \langle 0.4,0.8,0.3\rangle & \langle 0.5,0.4,0.3\rangle & \langle 0.3,0.4,0.5\rangle \\
\langle 0.2,0.3,0.6\rangle & \langle 0.3,0.7,0.2\rangle & \langle 0.5,0.4,0.3\rangle & \langle 0.2,0.6,0.5\rangle
\end{bmatrix}
\end{array}$$

Step 2 & **Step 3**

For c_1, we determine the following:

$$\begin{aligned}
\tau\left(\partial_1\right)&=\varpi_1\sup\left(\partial_1,\partial_2\right)+\varpi_1\sup\left(\partial_1,\partial_3\right)+\varpi_1\sup\left(\partial_1,\partial_4\right)\\
&=\varpi_1\left[2-d\left(\partial_1,\partial_2\right)\right]+\varpi_1\left[2-d\left(\partial_1,\partial_3\right)\right]+\varpi_1\left[2-d\left(\partial_1,\partial_4\right)\right]\\
&=6\varpi_1-\varpi_1\left[d\left(\partial_1,\partial_2\right)+d\left(\partial_1,\partial_3\right)+d\left(\partial_1,\partial_4\right)\right]\\
&=6\times 0.3-0.3\left[\begin{array}{l}\dfrac{\left|0.3^2-0.4^2\right|+\left|0.4^2-0.5^2\right|+\left|0.5^2-0.2^2\right|}{2}+\\ \dfrac{\left|0.3^2-0.1^2\right|+\left|0.4^2-0.5^2\right|+\left|0.5^2-0.3^2\right|}{2}+\\ \dfrac{\left|0.3^2-0.2^2\right|+\left|0.4^2-0.5^2\right|+\left|0.5^2-0.6^2\right|}{2}\end{array}\right]\\
&=1.6576
\end{aligned}$$

$$\begin{aligned}
\tau\left(\partial_2\right)&=\varpi_2\sup\left(\partial_2,\partial_1\right)+\varpi_2\sup\left(\partial_2,\partial_3\right)+\varpi_1\sup\left(\partial_2,\partial_4\right)\\
&=1.099
\end{aligned}$$

$$\tau(\partial_3) = \varpi_3 \sup(\partial_3,\partial_1) + \varpi_3 \sup(\partial_3,\partial_2) + \varpi_3 \sup(\partial_3,\partial_4)$$
$$= 0.55$$
$$\tau(\partial_4) = \varpi_4 \sup(\partial_4,\partial_1) + \varpi_4 \sup(\partial_4,\partial_2) + \varpi_4 \sup(\partial_4,\partial_3)$$
$$= 2.202$$

In a similar manner we obtain these values for c_2, c_3, c_4 and for the sake of simplicity we have listed them in a tabular form given by:

	$\tau(\partial_1)$	$\tau(\partial_2)$	$\tau(\partial_3)$	$\tau(\partial_4)$
c_1	1.6576	1.099	0.55	2.202
c_2	1.6125	1.025	0.553	2.088
c_3	1.528	1.089	0.524	2.094
c_4	1.575	0.99	0.534	2.144
c_5	1.545	1.03	0.517	2.132

Step 4 Utilizing the above 5 sets of values and the PNHPWA operator defined in definition **3.1.4**, we obtain the aggregate preference values of each IC, $c_i\ (i = 1,2,3,4,5)$ and they are denoted by $\tau(c_i)$. Suppose $\rho = 3$ then the preference values are given by:

$\tau(c_1) = (0.561, 0.647, 0.478)$,

$\tau(c_2) = (0.03, 0.873, 0.227)$,

$\tau(c_3) = (0.325, 0.298, 0.165)$,

$\tau(c_4) = (0.81, 0.437, 0.302)$ and

$\tau(c_5) = (0.468, 0.729, 0.286)$

Step 5 Using the definition **2.1.4,** we obtain the scores of each $\tau(c_i)$ denoted by $\$(\tau(c_i)), i = 1,2,3,4,5$

$\$(\tau(c_1))$=0.501, $\$(\tau(c_2))$=0.56, $\$(\tau(c_3))$=0.38, $\$(\tau(c_4))$=0.58, and $\$(\tau(c_5))$=0.55

Now rank all the insurance companies' performance according to their corresponding scores $\$(\tau(c_i)), i = 1,2,3,4,5$ we have, $\$(\tau(c_4)) \succ \$(\tau(c_2)) \succ \$(\tau(c_5)) \succ \$(\tau(c_1)) \succ \$(\tau(c_3))$ i.e.

$c_3 \prec c_1 \prec c_5 \prec c_2 \prec c_4$. Therefore, the best-performing insurance company preferred by the decision-maker is c_4. Thus, c_4 is the most desirable alternative based on the prescribed alternatives suggested by the decision-maker.

Step 6 End

Note: It is to be noted that, there should not be any change in the order of the rankings of the alternatives if we assign any value of $\rho > 1$.

5.1. Comparative analysis

Now a natural question arises that is there any changes in the order of the rankings of the alternatives or it will remain the same if we utilize the PNHPWG operator defined in definition **3.2.3**. To get an exact idea, we find out the following:

We repeat up to step 3

Step 4 If we use the PNHPWG operator defined in definition **3.2.3** then obtain the results given by:

$\tau^{/}(c_1) = (0.005439, 0.117, 0.260)$

$\tau^{/}(c_2) = (0.00948, 0.012, 0.1484)$

$\tau^{/}(c_3) = (0.01353, 0.00919, 0.099)$

$\tau^{/}(c_4) = (0.00827, 0.0105, 0.248)$

And

$\tau^{/}(c_5) = (0.0096, 0.0131, 0.2404)$

Step 5 Score values of the aggregates are given by

$\$(\tau^{/}(c_1))$=0.3153, $\$(\tau^{/}(c_2))$=0.3260, $\$(\tau^{/}(c_3))$=0.3301, $\$(\tau^{/}(c_4))$=0.3128, and $\$(\tau^{/}(c_5))$=0.31415

Clearly, $\$(\tau^{/}(c_4)) \prec \$(\tau^{/}(c_5)) \prec \$(\tau^{/}(c_1)) \prec \$(\tau^{/}(c_2)) \prec \$(\tau^{/}(c_3))$

Thus, c_3 is the optimal choice for the decision-maker

Comparisons based on two operators are given as:

Operator	Ranking	Optimal choice
PNHPWA	$c_3 \prec c_1 \prec c_5 \prec c_2 \prec c_4$	c_4
PNHPWG	$c_4 \prec c_5 \prec c_1 \prec c_2 \prec c_3$	c_3

Therefore, one can invest his money either in c_4 or c_3 to gain more in the future.

6. Conclusion

The present paper is devoted to constructing a MADM model based on Pythagorean neutrosophic Hamacher aggregate operators to deal with real-world problems. We also propose the new scores and accuracy functions to compare PNNs. A practical application has been successfully executed to show the effectiveness of the proposed model. At present, the Pilthogenic set [53] is considered to be the most generalized framework to model vagueness. Therefore, in the future, the proposed study can be extended in Pilthogenic setting by introducing the Pythagorean Pathogenic Hamacher operators and studying its various weightage operators that will help to tackle more uncertain knowledge hidden in our surroundings.

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