REVISTAS CIENTÍFICAS INDEXADAS BISTUA
FACULTAD DE CIENCIAS BÁSICAS - UNIVERSIDAD DE PAMPLONA

BISTUA *Rev. FCB, Volumen 22 (1) (2024). Pamplona-Colombia.*
https://doi.org/10.24054/bistua.v22i1.2551

Another look at the solutions of the modified Benjamin-Bona-Mahony equation and the modified Korteweg-de Vries equation

Otro vistazo a las soluciones de la ecuación de Benjamin-Bona-Mahony modificada y la ecuación de Korteweg-de Vries modificada

Juan Carlos López[a,b]; **Rosalba Mendoza Suárez**[b]; **Jairo Mendoza Suárez**[c]

[a] *Programa Matemáticas, Universidad de Pamplona, Colombia; jclopez@unipamplona.edu.co*
[b] *Programa Matemáticas, Universidad de Pamplona, Colombia; rosalbame@unipamplona.edu.co*
[c] *Programa Física, Universidad de Pamplona, Colombia; jairoam@unipamplona.edu.co*

Autor Corresponsal: *jairoam@unipamplona.edu.co*

Recibido*: Septiembre 18, 2023.* ***Aceptado****: Junio 20, 2024.* ***Publicado****: Junio 21, 2024*

Resumen

Islam MS, Khan K, Akbar MA, Mastroberardino A., presentan el método F-expansión combinado con la ecuación de Riccati para resolver ecuaciones de evolución no lineal, y lo aplican para encontrar "*nuevas soluciones*", de las ecuaciones $mBBM$ y $mKdV$. En esta nota demostramos que estas soluciones se pueden encontrar a partir de la solución general y que el método expuesto por los citados autores, no es tan efectivo como ellos afirman.

Palabras clave: Soluciones de onda viajera, Ecuaciones diferenciales parciales, Física matemática.

Abstract:

Islam MS, Khan K, Akbar MA, Mastroberardino A., present the F-expansion method combined with the Riccati equation to solve non-linear evolution equations, and apply it to find '*new solutions* of the $mBBM$ and $mKdV$ equations. In this note, we demonstrate that these solutions can be found from the general solution and that the method proposed by the aforementioned authors is not as effective as they claim

Keywords: Traveling wave solutions, Partial differential equations, Mathematical physics.

1. Introducción

Exact solutions of nonlinear partial differential equations play an important role in physical sciences, as these equations describe various natural phenomena like vibrations, solitons, wave propagation, etc. (see [1–4]). Thus these solutions can give us a better understanding of the physical aspects of the problem.

In recent years, with the development of computer algebraic systems, various methods have been implemented to find traveling wave solutions, such as the hyperbolic tangent method, the exponential method, the Riccati equation projective method, etc. (see [5–8]).

In [9] the authors present an analytical method, called the improved F-expansion method combined with a Riccati equation, to find exact solutions of evolution equations. To verify computational efficiency, the proposed method is applied to find solutions of the following partial differential equations:

The modified Benjamin-Bona-Mahony equation (mBBM)

$$u_t + u_x + au^2u_x + bu_{xxt} = 0, \tag{1}$$

with a, b positive constants, and the modified Korteweg-de Vries equation (mKdV)

$$u_t - u^2u_x + \delta u_{xxt} = 0, \tag{2}$$

with δ a non-zero constant. For the above equations, look for solutions in the form of a solitary wave

$$u(x,t) = u(\xi), \qquad \text{where} \qquad \xi = x + \lambda t.$$

By substituting into (1), integrating the resulting equation with respect to ξ, and taking the constant of integration equal to zero, they obtain the ordinary differential equation.

$$b\lambda u'' + \frac{a}{3}u^3 + (\lambda + 1)u = 0. \tag{3}$$

In a similar way they obtain for the equation (2), the equation

$$\delta u'' - \frac{1}{3}u^3 + \lambda u = 0. \tag{4}$$

It can be seen that the equations (3) and (4) are of the form

$$u'' + Au^3 + Bu = 0 \tag{5}$$

with A and B constant.

The purpose of this note is to show that the equations (3) and (4) can be solved using elementary methods, and that the 58 "solutions" presented in [9], they are obtained as particular cases of the "general solution" found in this note; so that the solutions found in [9] should not be considered as new and that the method used by the authors of [9] is not as effective as they claim in the conclusions.

2. Solution to the equation (5)

Following the suggestion found in basic texts on differential equations, see [10] substitutions

$$u'(\xi) = \rho$$
$$u''(\xi) = \frac{d\rho}{d\xi} = \frac{d\rho}{du}\frac{du}{d\xi} = \rho\frac{d\rho}{du} \tag{6}$$

allow to reduce the equation (5) to the first order differential equation

$$\rho\frac{d\rho}{du} = -Au^3 - Bu, \tag{7}$$

from where,

$$\rho = \pm\sqrt{-\frac{A}{2}u^4 - Bu^2 + C_1}, \tag{8}$$

Taking into account the value of ρ, given in (6) and separating the variables, we obtain

$$\int \frac{du}{\sqrt{-\frac{A}{2}u^4 - Bu^2 + C_1}} = \pm\xi + \tilde{C}_2 \tag{9}$$

C_1 y $\tilde{C}_2$ arbitrary constants.

As is known the solution of (9) can be expressed via the elliptic Weierstrass functions, see [11]. However, periodic solutions and solitary wave solutions can be found, for some particular values of the constants C_1 and $\tilde{C}_2$.

1. If $C_1 = 0$, we obtain the following solutions according to the signs of the constants A and B.

   a) If $A < 0$ and $B > 0$,
$$u(\xi) = \sqrt{\frac{-2B}{A}}\sec\left[\pm\sqrt{B}\xi + C_2\right], \tag{10}$$

   b) If $A < 0$ and $B < 0$,
$$u(\xi) = 2\sqrt{\frac{2B}{A}}\quad\frac{C_2\exp[\pm\sqrt{-B}\xi]}{C_2^2\exp[\pm 2\sqrt{-B}\xi] - 1}, \tag{11}$$

   c) If $A > 0$ and $B < 0$,
$$u(\xi) = \pm 2\sqrt{\frac{-2B}{A}}\quad\frac{C_2\exp[\pm\sqrt{-B}\xi]}{C_2^2\exp[\pm 2\sqrt{-B}\xi] + 1}, \tag{12}$$

   d) If $A < 0$ and $B = 0$,
$$u(\xi) = \pm\sqrt{\frac{-2}{A}}\frac{1}{\xi + C_2}, \tag{13}$$

2. If $C_1 = -\frac{B^2}{2A}$, the following solutions are obtained

   a) For $A < 0$ and $B < 0$
$$u(\xi) = \sqrt{\frac{B}{A}}\tan\left[\pm\sqrt{-\frac{B}{2}}\xi + C_2\right], \tag{14}$$

   b) For $A < 0$ and $B > 0$
$$u(\xi) = \pm\sqrt{\frac{-B}{A}}\quad\frac{1 + C_2\exp[\pm\sqrt{2B}\xi]}{1 - C_2\exp[\pm\sqrt{2B}\xi]}, \tag{15}$$

Note: the other cases that may arise regarding the values that the constants A and B can assume are not considered here, since they are not relevant to comment on the solutions of the equations in [9].

3. The modified Benjamin-Bona - Mahony equation

The application of the method described by the authors of [9], it involves taking $u(\xi)$ defined by

$$u(\xi) = \alpha_0 + \alpha_1(m + F(\xi)) + \beta_1(m + F(\xi))^{-1},$$

by substituting $u(\xi)$ and its second derivative into (3), we obtain a polynomial in $F(\xi)$. Setting the coefficients of the powers of $F(\xi)$ to zero, we obtain a system of seven equations in the unknowns α_0, α_1, β_1 m_1, λ. (see page 4 in [9]), due to the complexity of the system, it is clear the need to use a computer algebraic system to find its solutions.
The objective of this section is to comment on the 58 solutions of the equation (3), reported by the authors of the article under study and to observe that they are obtained from the equations given in the previous section for particular values of the constants of integration.

We first make the following observations:

1. If $u(\xi)$ is a solution of the equation (3), then $v(\xi) := -u(\xi)$ is also a solution; therefore, it is only necessary to "examine.ªbout half of the solutions.

2. A direct observation of the *families 1 and 2* of solutions, taking into account that the value of ξ is the same, it is obtained that: $u_5 = u_4$; $u_6 = u_3$; $u_8 = u_1$; $u_7 = u_2$; that is, the solutions of *family 2* are the same as those of *family 1*.

3. The solutions of *family 1* are obtained from the solutions of *family 6*, taking $\alpha_0 = 0$.

4. In *family 3*, $u_9 = u_{11}$; $u_{10} = u_{12}$; for which it is enough to take into account that by definition
$$\coth\alpha = \frac{1}{\tanh\alpha}$$

5. The solutions of the *family 5,* are obtained from the *family 6*, with
$$\alpha_0 = \frac{1}{\sqrt{-a}}.$$

6. The solutions of the *families 7 and 8* are obtained from those of the *family 12,* taking $\alpha_0 = 0$.

7. In the *family 9*, $u_{35} = u_{33}$; $u_{36} = u_{34}$;

8. If in the *family 12,* we take $\alpha_0 = \frac{i}{\sqrt{a}}$ we get the *family 11.*

9. The solutions of the *family 15*, are the same of the *family 13*.

10. The expressions given in the *family 14* are not solutions.

11. The solutions of the *family 16,* are obtained from those of the *family 17,* with

$$\alpha_0 = \frac{1}{2\sqrt{6b}}.$$

From the above observations, it is clear that it is only necessary to consider the following ten solutions $u_9, u_{13}, u_{15}, u_{21}, u_{35}, u_{37}, u_{39}, u_{49}, u_{57}, u_{58}$.

Taking into account that (3), it can be written as

$$u'' + \frac{a}{3b\lambda}u^3 + \frac{\lambda+1}{b\lambda}u = 0,$$

which corresponds to the equation (5), with

$$A = \frac{a}{3b\lambda}; \quad B = \frac{\lambda+1}{b\lambda} \tag{16}$$

Using the expressions for $\tanh$ and $\coth$ in terms of the exponential function, the solution $u_9(\xi)$ can be written:

$$u_9(\xi) = \frac{-2\sqrt{-6bk}}{\sqrt{a(1+8bk)}} \cdot \frac{1+\exp[-4\sqrt{-k}\xi]}{1-\exp[-4\sqrt{-k}\xi]},$$

For this solution $\lambda = -\frac{1}{1+8bk}$, plugging this value of λ into the expressions for A and B given in the equation (16). It is observed that the equation $u_9(\xi)$ is obtained from the equation (15), with $C_2 = 1$.
Proceeding in the same way, the following can be stated:

$$u_{13}(\xi) = \frac{-4\sqrt{-6bk}}{\sqrt{a(1-4bk)}} \cdot \frac{\exp[2\sqrt{-k}\xi]}{\exp[4\sqrt{-k}\xi]-1}, \text{it is (11) with } C_2 = 1$$

$$u_{15}(\xi) = \frac{-4\sqrt{-6bk}\; i}{\sqrt{a(1-4bk)}} \cdot \frac{\exp[2\sqrt{-k}\xi]}{\exp[4\sqrt{-k}\xi]+1}, \text{it is (11) with } C_2 = i.$$

$$u_{21}(\xi) = -\frac{\sqrt{-6bk}}{\sqrt{a(1+2bk)}} \cdot \frac{1+C_2\exp[-2\sqrt{-k}\xi]}{1-C_2\exp[-2\sqrt{-k}\xi]}, \text{it is (15).}$$

For the solution $u_{35}(\xi)$ given in [9]:

$$u_{35}(\xi) = \frac{-b\sqrt{6k}}{\sqrt{ab(1+8bk)}} \left[(\cot^2\sqrt{k}\xi) - 1\right]\tan(\sqrt{k}\xi),$$

taking into account some basic trigonometric identities

$$\cot\alpha - \tan\alpha = \frac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha} = \frac{2\cos(2\alpha)}{\sin(2\alpha)} = 2\cot 2\alpha$$

Thus u_{35} can be written

$$u_{35}(\xi) = \frac{-2b\sqrt{6k}}{\sqrt{ab(1+8bk)}}\cot(2\sqrt{-k}\xi),$$

in this solution $\xi = x - \frac{t}{1+8bk}$; that is $\lambda = \frac{1}{1+8bk}$, by substituting this value of λ in the expressions for A and B given in (16), it is observed that $u_{35}(\xi)$ is obtained from the equation (14) with $C_2 = \frac{\pi}{2}$.
Similarly, we have:

$$u_{37}(\xi) = \frac{-2i\sqrt{6bk}}{\sqrt{a(4bk-1)}}\csc(2\sqrt{k}\xi),$$

is obtained from (10), with $C_2 = \frac{\pi}{2}$

$$u_{39}(\xi) = \frac{2i\sqrt{6bk}}{\sqrt{a(4bk-1)}}\sec(2\sqrt{k}\xi),$$

is obtained from (10), with $C_2 = 0$.

Finally,

$$u_{49}(\xi) = -\frac{\sqrt{6b}}{\sqrt{a}\xi}, \quad \text{is (13), with } C_2 = 0.$$

$$u_{57}(\xi) = \frac{-6\alpha_0 b}{\sqrt{6ab}\;\alpha_0\;\xi - 6b}, \quad \text{is (13), with } C_2 = \frac{\sqrt{6b}}{\alpha_0}.$$

$$u_{58}(\xi) = \frac{6\alpha_0 b}{\sqrt{6ab}\;\alpha_0\;\xi + 6b}, \quad \text{is (13), with } C_2 = \frac{\sqrt{6b}}{\sqrt{a}\alpha_0}.$$

4. The modified Korteweg-de Vries (mKDV) equation

In example 4.2 of reference [9], the authors apply the method they describe to the equation mKDV, given in (2) and to do so they solve the equation (4) which is of the form (5) with:

$$A = -\frac{1}{3\delta}; \quad B = \frac{\lambda}{\delta}. \tag{17}$$

The application of the "improved F-expansion method combined with the Riccati equation"to solve the equation mKDV, also leads to the resolution of a system of 7 equations in the unknowns $\alpha_0, \alpha_1, \beta_1, m, \lambda$ system that they solve with the help of mathematical software. This allows them to find 34 solutions of the equation (4).

The objective of this section is to show that the solutions found by the authors in [9] are obtained as particular cases of the solutions found in this article in section 3. Before doing so, we make the following comments:

1. If $u(\xi)$ is a solution of the equation (4) then $v(\xi) = -u(\xi)$ is also a solution.

2. The solutions of families 1 and 3 are obtained from those corresponding to family 2, taking $\alpha_0 = 0$.

3. The value of λ must be $\lambda = -2\delta k \pm 6\delta k$, with $k < 0$, for the solutions of family 4; with $k > 0$ for family 8.

4. The solutions of families 5 and 7 are obtained from the solutions of family 6, with $\alpha_0 = 0$.

5. The solutions of family 11 are the same as those of family 10.

6. As in family 2, α_0 is an arbitrary constant, the solutions $u_{7,8}(\xi)$ are obtained from the expressions of $u_{5,6}(\xi)$. Similarly, the solutions of $u_{21,22}(\xi)$ corresponding to family 6 are obtained from $u_{19,20}(\xi)$.

Considering the above, it is only necessary to look at the solutions, $u_5, u_6, u_{13}, u_{19}, u_{20}, u_{27}, u_{29}, u_{30}, u_{31}$.

Starting with some observations of the solution $u_{27}(\xi)$. On page 10 of the article [9] it appears:

$$Family\ 8: \quad u_{27}(\xi) = \sqrt{6\delta k}\csc[\sqrt{k}\xi]\sec[\sqrt{k}\xi],$$

where $\xi = x + (-2\delta k \pm 6\delta)t$. Possibly due to a typing error, the value of λ, as can be seen on page 8, should be $\lambda = -2\delta k \pm 6\delta k$, with which λ can take two values $\lambda_1 = 4\delta k$ and $\lambda_2 = -8\delta k$. However, a direct calculation shows that $u_{27}(\xi)$ is not a solution of the equation (4) with $\lambda_2 = -8\delta k$. Thus, the only value of λ must be the one given in $\lambda_1 = 4\delta k$, with $k > 0$.

Using basic trigonometric identities, we can simplify the expression for $u_{27}(\xi)$:

$$\begin{aligned} u_{27}(\xi) &= \sqrt{6\delta k}\csc[\sqrt{k}\xi]\sec[\sqrt{k}\xi] = \frac{\sqrt{6\delta k}}{\sin[\sqrt{k}\xi]\cos[\sqrt{k}\xi]} \\ &= \frac{2\sqrt{6\delta k}}{2\sin[\sqrt{k}\xi]\cos[\sqrt{k}\xi]} = \frac{2\sqrt{6\delta k}}{\sin[2\sqrt{k}\xi]} \\ &= 2\sqrt{6\delta k}\csc[2\sqrt{k}\xi] \end{aligned}$$

On the other hand, taking into account the values of A and B given in (4.1)

$$A = \frac{1}{3\delta} < 0; \quad B = \frac{\lambda}{\delta} = 4k > 0,$$

It can be seen that $u_{27}(\xi)$ is obtained from equation (2.5) with $C_2 = -\pi/2$.

A similar analysis can be done for the solutions of family 4, in this case $\lambda = 4\delta k$.

$$u_{13}(\xi) = 4\sqrt{-6k\delta}\,\frac{\exp[-2\sqrt{-k}\xi]}{1-\exp[-4\sqrt{-k}\xi]},$$

is obtained from (2.6) with $C_2 = -1$.

For the remaining solutions we have:

1.
$$u_5(\xi) = \sqrt{-6k\delta}\,\frac{1 - C\exp[2\sqrt{-k}\xi]}{1 + C\exp[2\sqrt{-k}\xi]}, \quad \text{with} \quad C = \frac{\alpha_0 - \sqrt{-6k\delta}}{\alpha_0 + \sqrt{-6k\delta}},$$

   so; $u_5(\xi)$ is equation (2.10) with $C_2 = -C$.

   In the case $\alpha_0 = -\sqrt{-6k\delta}$, $u_5(\xi) = -\sqrt{-6k\delta}$ is obtained from (2.10) with $C_2 = 0$.

2. After simplifying the expression for $u_{20}(\xi)$ we have:

$$u_{20}(\xi) = -\sqrt{-6k\delta}\left(\frac{\tan[\sqrt{k}\xi] + \frac{\sqrt{-6k\delta}}{\alpha_0}}{1 - \frac{\sqrt{-6k\delta}}{\alpha_0}\tan[\sqrt{k}\xi]}\right)$$

   is obtained from equation (2.9) by taking
   $C_2 = \tan^{-1}\left[\frac{\sqrt{6k\delta}}{\alpha_0}\right]$.
   If $\alpha_0 = 0$,

$$u_{20}(\xi) = \sqrt{-6k\delta}\cot[\sqrt{k}\xi],$$

   in this case $u_{20}(\xi)$ is (2.9) with $C_2 = \pi/2$.

3.
$$u_{29,30}(\xi) = \sqrt{6\delta}\,\frac{1}{\xi \mp \frac{\sqrt{6\delta}}{\alpha_0}},$$

   is (2.8) with $C_2 = \mp\frac{\sqrt{6\delta}}{\alpha_0}$,

$$u_{31}(\xi) = -\frac{\sqrt{6\delta}}{\xi}$$

   is (2.8) with $C_2 = 0$.

5. Conclusion

It was shown that some "solutions.ºf the equations $mBBM$ and $mKdV$ found in [9] are not solutions, despite what the authors of the article under study stated, according to which: "*All of these solutions have been verified with MAPLE by substituting them into the original equations.*"
In this note it was shown that the solutions found by the authors of [1] are obtained from the general solution, found in section 2 of this article, for particular values of the constants.
Finally the authors of [9] conclude: "*the performance on the improved F. expansion method confirms that it is a reliable an effective technique for finding exact solution ...*". However, what is shown in this note tells us that, at least for the equations considered by them, the method is not as effective.

Referencias

[1] C.A. Gómez, Harun-Or Roshid, M. Inc, L. Akinyemi, and H. Rezazadeh. (2022). On soliton solutions for perturbed Fokas- Lenells eqution. *Optical and Quantum Electronics*, vol. 54, no. 370, 1–10.

[2] K. S. A. Ghafri and E. V. Krishrar. (2020). Arjar Biswas, Chirped optical soliton perturbation of Fokas-Lenells equation with full nonlinearity. *Advanced in Differential Equations*. vol. 2020, 191.

[3] A. Bansal, A. H. Kara, A. Biswas, S. P. Moshokoa, and M. Belic. (2018). Optical soliton perturbation, group invariants and conservation laws of perturbed Fokas–Lenells equation. *Chaos, Solitons and Fractals*, vol. 114, 275–280.

[4] K. K. Ali, M. S. Osman, and M. Abdel Aty. (2020). New optical solitary wave solutions of Fokas-Lenells equation in optical :ber via Sine-Gordon expansion method. *Alexandria Engineering Journal*, vol. 59, no. 3, 1191–1196.

[5] Y. Yildirim, A. Biswas, M. Asma et al.(2018). Chirp-free bright optical soliton perturbation with Chen–Lee–Liu equation by traveling wave hypothesis and semi-inverse variational principle. *Optik*, vol. 172, no. 172, 772–776.

[6] E. M. E. Zayed and K. A. E. Alurr. (2016). New extended auxiliary equation method and its applications to nonlinear Schrödinger-type equations. *Optik*, vol. 127, no. 20, 9131–9151, 2016.

[7] A. M. Wazwaz.(2007). The tanh-coth method for solitons and kink solutions for nonlinear parabolic equations. *Applied Mathematics and Computation*, vol. 188, no. 2, 1467–1475.

[8] S. B. G. Karakoc, K. K. Ali, and M. S. Mehanna. (2023). Exact traveling wave solutions of the schamel-KdV equation with two different methods. *Universal Journal of Mathematics and Applications*, vol. 6, no. 2, 65–75.

[9] Islam MS, Khan K, Akbar MA, Mastroberardino A. (2014). A note on improved F-expansion method combined with Riccati equation applied to nonlinear evolution equations. *R.Soc. open sci*. 1: 140038. 1-13. https://royalsocietypublishing.org/doi/epdf/10.1098/rsos.140038.

[10] Edwards, Penney. (1986). Ecuaciones diferenciales con aplicaciones México. *Prentice-Hall*.

[11] Whittaker, E.T, Watson.G.V. (2008). A course of Modern Analysis. *fourth edition. Cambridge. Cambridge University Press*. http://dx.doi.org/10.1007/978-94-010-0273-8-11.