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Equivalent kernels, dual relation, and applications

Núcleos equivalentes, relación dual y aplicaciones

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Resumen

El objetivo de este trabajo es estudiar algunos núcleos definidos positivos a valores operadores en espacios de Hilbert. Demostramos la existencia de un núcleo K_2 asociado a cualquier par de núcleos equivalentes K_1 y K. El par (K_1, K_2) se llama núcleos biequivalentes. Además, demostramos que K_2 y K son equivalentes y satisfacen una relación dual similar a las bases de Riesz, las sucesiones biortogonales y los marcos duales en los espacios de Hilbert. Como una consecuencia, obtenemos nuevos resultados para los procesos estocásticos.

Palabras clave: Nucleos definidos positivos; Sistemas biortogonales; Descomposición de Kolgomorov; Kernels biequivalentes; Relación dual.

Abstract:

The aim of this paper is to study some positive definite kernels for operator values in Hilbert spaces. We prove the existence of a kernel K_2 associated with any pair of equivalent kernels K_1 and K. The pair (K_1, K_2) is called biequivalent kernels. Moreover, we show that K_2 and K are equivalent and satisfy a dual relation similar to Riesz bases, biorthogonal sequences, and dual frames in Hilbert spaces. As a consequence, we obtain new results for stochastic processes.

Keywords: Positive definite kernels; biorthogonal systems; Kolgomorov decomposition; biequivalent kernels; dual relation.

1 Introduction

In this paper, we study positive definite kernels for operator values in Hilbert spaces. The positive definite kernels play an increasingly prominent role in many applications such as scattered data fitting, numerical solution of PDEs, probability theory and statistics, and stochastic analysis. Extensions for kernels are given to the operator values of the results obtained in [2] for the scalar case. We obtain results are similar with some known results about biorthogonal bases, dual frames, and wavelets. These results are motivated by the importance and applications of dual bases, frames dual, and wavelets and their great relevance in pure and applied mathematics, see [6, 7, 8, 11, 10].

Let $(X, \|\cdot\|)$ be a Banach space. Let I be nonempty. A familiy $\{(x_i, x_i^*)\}_{i\in I}$ of pairs in $X \times X^*$ is called a *biorthogonal system* in $X \times X^*$ if

$$\langle x_i, x_i^* \rangle = \delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker δ, for all $i, j \in I$.
Let T be a positive linear operator on the Hilbert space $\mathcal{H}$ such that for every $x \in \mathcal{H}$

$$A\langle x, x\rangle \le \langle Tx, x\rangle \le B\langle x, x\rangle \tag{1.1}$$

for some $0 < A \le B$. Then T is invertible on $\mathcal{H}$ and for every $x \in \mathcal{H}$ it follows that

$$\frac{1}{B}\langle x, x\rangle \le \left\langle T^{-1}x, x\right\rangle \le \frac{1}{A}\langle x, x\rangle .$$

We now recall the definition of a Riesz basis (see [11]). Let $\mathcal{H}$ be a separable Hilbert space. A family $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is called a Riesz basis if it is the image of an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ under an bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$, that is, if

$$Te_n = x_n \quad \text{for} \quad (n = 1, 2, 3, ...) .$$

There are numerous alternative but equivalent definitions of a Riesz basis (see [11], Theorem 9 on page 32).
Let $\mathcal{H}$ be a Hilbert space. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ if and only if the sequence $\{x_n\}_{n\in\mathbb{N}}$ is complete

in $\mathcal{H}$ and there exist positive constants A, B, such that for any sequence of scalars $a = \{a_n\}_{n\in\mathbb{N}}$ with finite support one has

$$A\sum_{n\in\mathbb{N}}|a_n|^2 \leq \left\|\sum_{n\in\mathbb{N}} a_n x_n\right\|^2 \leq B\sum_{n\in\mathbb{N}}|a_n|^2.$$

Proposition 1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Riesz basis for a Hilbert space $\mathcal{H}$ and let $\{y_n\}_{n\in\mathbb{N}}$ be biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$. Then for every $x \in \mathcal{H}$ there exist positive constants A, B such that*

$$A\sum_{n\in\mathbb{N}} |\langle x, y_n\rangle|^2 \leq \|x\|^2 \leq B\sum_{n\in\mathbb{N}} |\langle x, y_n\rangle|^2 .$$

The following dual relation holds

$$\frac{1}{B}\sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq \|x\|^2 \leq \frac{1}{A}\sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 .$$

Definition 2. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, or simply, a g- frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i\in I}\|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A = B$ and we call it a Parseval g-frame if $A = B = 1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in I}\Lambda_i^*\Lambda_i f, \quad f \in \mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \leq S_\Lambda \leq BI$$

and for each $f \in \mathcal{H}$, we have

$$\frac{1}{B}\|f\|^2 \leq \sum_{i\in I}\left\|\Lambda_i S_\Lambda^{-1} f\right\|^2 \leq \frac{1}{A}\|f\|^2 .$$

The canonical dual g-frame for Λ is defined by $\left\{\Lambda_i S_\Lambda^{-1}\right\}_{i\in I}$ with bounds $\frac{1}{B}, \frac{1}{A}$.

Our paper is organized as follows. After an introduction, Section 2 presents some basic definitions concerning positive operators, Riesz bases, biorthogonal systems, g-frames in Hilbert spaces and Kolmogorov decompositions that are used throughout this paper, see for instance [4, 11, 9, 3]. Sections 3 present some results about positive definite kernels for operator values in Hilbert spaces, see [1]. We show that two kernels K_2 and K are equivalent and satisfy a dual relation similar to Riesz bases, biorthogonal sequences, and dual frames in Hilbert spaces. Finally, section 4, we obtain new results for stochastic processes.

2 Preliminaries

In this section, we describe some properties of bases in Banach spaces, Hilbert spaces and linear operators, and Kolmogorov Decomposition Theorem. For more details, see instance [4, 11, 3].

Definition 3. Let $(X, \|\cdot\|)$ be a Banach space. Let I be a nonempty. A familiy $\{(x_i, x_i^*)\}_{i\in I}$ of pairs in $X \times X^*$ is called a *biorthogonal system* in $X \times X^*$ if

$$\langle x_i, x_i^*\rangle = \delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker δ, for all $i, j \in I$. From now on, we deal only with the case that the index set I is countable.

Given a basis $\{x_n\}_{n\in I}$, we define the coordinate functionals $x_n^* \colon X \to \mathbb{R}$ by $x_n^*(x) = a_n$, where $x = \sum_{n=1}^{\infty} a_n x_n$. It is easily seen that each x_n^* is linear and satisfies $x_n^*(x_m) = \delta_{m,n}$ for all $n, m \in I$. .

Definition 4. Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{C}$ and $\{(x_i, x_i^*)\}_{i\in I}$ be a biorthogonal system. Then

(i) The system $\{x_i\}_{i\in I}$ is *total* if $\langle x, x_i^*\rangle = 0$ for all $i \in I$ implies $x = 0$.

(ii) The system $\{x_i\}_{i\in I}$ is *fundamental* if the finite linear combinations of $\{x_i\}_{i\in I}$ are dense in X. i.e. if $\langle x_i, x^*\rangle = 0$ for all $i \in I$ implies $x^* = 0$,

(iii) The system $\{x_i\}_{i\in I}$ is *bounded* if there exists a constant $c \geq 1$ such that $\|x_i\|\,\|x_i^*\| \leq c$ for all $i \in I$.

(iv) A biorthogonal system $\{(x_i, x_i^*)\}_{i\in I}$ is said to be *complete* if it is fundamental and total.

2.1 Positive Operators, Riesz Bases, Biorthogonal Systems in Hilbert Spaces and g-frames

Next, we summarize some well-known results about positive operators, Riesz bases, biorthogonal systems, and g-frames (see [4], [11] and [9]) .

Definition 5. A bounded operator T acting on a Hilbert space $\mathcal{H}$ is said to be positive if $T = T^*$ and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

Proposition 6. *[4, Theorem 4.6.11] Let T be a positive linear operator on the Hilbert space $\mathcal{H}$ such that for every $x \in \mathcal{H}$*

$$A\langle x, x\rangle \leq \langle Tx, x\rangle \leq B\langle x, x\rangle, \tag{2.1}$$

for some $0 < A \leq B$. Then T is invertible on $\mathcal{H}$ and for every $x \in \mathcal{H}$ it follows that

$$\frac{1}{B}\langle x, x\rangle \leq \left\langle T^{-1}x, x\right\rangle \leq \frac{1}{A}\langle x, x\rangle . \tag{2.2}$$

In a separable Hilbert space, the most important bases are orthonormal. Second in importance are those bases that are equivalent to some ortho-normal bases. They will be called Riesz bases, and they constitute the largest and most tractable class of bases known. Some definitions and results can be found, for example, in [11].

Definition 7. Let $\mathcal{H}$ be a separable Hilbert space. A family $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is called a Riesz basis if it is the image of an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ under an bounded invertible operator $T:\mathcal{H}\to\mathcal{H}$, that is, if

$$Te_n = x_n \quad \text{for} \quad (n=1,2,3,...).$$

Proposition 8. *Let $\mathcal{H}$ be a Hilbert space. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ if and only if the sequence $\{x_n\}_{n\in\mathbb{N}}$ is complete in $\mathcal{H}$ and there exist positive constants A, B, such that for any sequence of scalars $a=\{a_n\}_{n\in\mathbb{N}}$ with finite support one has*

$$A\sum_{n\in\mathbb{N}}|a_n|^2 \le \left\|\sum_{n\in\mathbb{N}} a_n x_n\right\|^2 \le B\sum_{n\in\mathbb{N}}|a_n|^2.$$

For the proof of this see [11, Theorem 9 on page 32].

Proposition 9. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Riesz basis for a Hilbert space $\mathcal{H}$ and let $\{y_n\}_{n\in\mathbb{N}}$ be biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$. Then for every $x\in\mathcal{H}$ there exist positive constants A, B such that*

$$A\sum_{n\in\mathbb{N}}|\langle x,y_n\rangle|^2 \le \|x\|^2 \le B\sum_{n\in\mathbb{N}}|\langle x,y_n\rangle|^2.$$

The following dual relation holds

$$\frac{1}{B}\sum_{n\in\mathbb{N}}|\langle x,x_n\rangle|^2 \le \|x\|^2 \le \frac{1}{A}\sum_{n\in\mathbb{N}}|\langle x,x_n\rangle|^2.$$

We have the following generalization of the previous result, see for example [9].

Definition 10. We call $\Lambda=\{\Lambda_i\in B(\mathcal{H},\mathcal{H}_i): i\in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \le B\|f\|^2, \quad f\in\mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A=B$ and we call it a Parseval g-frame if $A=B=1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in I}\Lambda_i^*\Lambda_i f, \quad f\in\mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \le S_\Lambda \le BI$$

and for each $f\in\mathcal{H}$, we have

$$\frac{1}{B}\|f\|^2 \le \sum_{i\in I}\left\|\Lambda_i S_\Lambda^{-1} f\right\|^2 \le \frac{1}{A}\|f\|^2.$$

The canonical dual g-frame for Λ is defined by $\left\{\Lambda_i S_\Lambda^{-1}\right\}_{i\in I}$ with bounds $\frac{1}{B}, \frac{1}{A}$.

2.2 Kolmogorov Decomposition Theorem

Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$. Then, the closure of the linear span of these spaces is denoted by $\bigvee_{n\in\mathbb{Z}}\mathcal{H}_n$. If the subspaces $\mathcal{H}_n$ are pairwise orthogonal, i.e. $\mathcal{H}_i\perp\mathcal{H}_j$ for $i\neq j$, then the notation $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ is used instead of $\bigvee_{n\in\mathbb{Z}}\mathcal{H}_n$. This space $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ will be called the orthogonal sum of the pairwise orthogonal subspaces $\mathcal{H}_n$.

Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of Hilbert spaces. An *operator- valued kernel on $\mathbb{Z}$ to $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$* is an application $K:\mathbb{Z}\times\mathbb{Z}\to\bigcup_{m,n\in\mathbb{Z}}\mathcal{L}(\mathcal{H}_m,\mathcal{H}_n)$ such that $K(n,m)\in\mathcal{L}(\mathcal{H}_m,\mathcal{H}_n)$ for $n,m\in\mathbb{Z}$.

In this section and the following one, unless it is otherwise stated, the kernels will be operator- valued ones.

A kernel K on $\mathbb{Z}$ with $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a *positive definite kernel* if

$$\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m,h_n\rangle_{\mathcal{H}_n}\ge 0,$$

for all sequences $\{h_n\}$ in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ with finite support(ie. $h_n=0$ except for finite number of integers n).

Let K be a positive definite kernel. Let $\mathcal{F}$ be the linear space of elements $\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n$ and $\mathcal{F}_o$ the space of elements $\mathcal{F}$ with finite support.

Define $B_K:\mathcal{F}_o\times\mathcal{F}_o\to\mathbb{C}$ with

$$B_K(f,g)=\sum_{m,n\in\mathbb{Z}}\langle K(n,m)f_m,g_n\rangle_{\mathcal{H}_n}, \tag{2.3}$$

for $f,g\in\mathcal{F}_o$, $f=\{f_n\}$, $g=\{g_n\}$, $f_n,g_n\in\mathcal{H}_n$.

Note that B_K, satisfies all properties of an inner product, except for the fact that the set

$$\mathcal{N}_K=\{h\in\mathcal{F}_o: B_K(h,h)=0\}$$

could be nontrivial.

According to the Cauchy-Schwarz inequality

$$\mathcal{N}_K=\{h\in\mathcal{F}_o: B_K(h,g)=0, \text{ for all } g\in\mathcal{F}_o\},$$

hence $\mathcal{N}_K$ is a linear subspace of $\mathcal{F}_o$.

The quotient space $\mathcal{F}_o/\mathcal{N}_K$ is also a linear subspace. If $[h]$ stands for the class in $\mathcal{F}_o/\mathcal{N}_K$ of the element h then the application

$$\langle[h],[g]\rangle = B_K(h,g), \qquad h,g\in\mathcal{F}_o$$

is well defined. To prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{F}_o/\mathcal{N}_K$ is straightforward.
The complection of $\mathcal{F}_o/\mathcal{N}_K$ with respect to the norm induced by this inner product is a Hilbert space. It is known as *the Hilbert space associated with the positive definite kernel* K and it is denoted by $\mathcal{H}_K$. The inner product and the norm of $\mathcal{H}_K$ will be represented as $\langle \cdot, \cdot \rangle_{\mathcal{H}_K}$ and $\| \cdot \|_{\mathcal{H}_K}$ respectively. This norm will be named as *the norm induced by* K.
The following theorem is a version of the classical result of Kolmogorov (See [5] for a historical review).

Theorem 11 (Kolmogorov)**.** *Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of Hilbert spaces and let $K : \mathbb{Z}\times\mathbb{Z} \to \bigcup_{m,n\in\mathbb{Z}} \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n)$ is a positive definite kernel . Then there exists an application V defined on $\mathbb{Z}$ such that $V(n) \in L(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ and*

(a) $K(n, m) = V^*(n)V(m)$ *if* $n, m \in \mathbb{Z}$.

(b) $\mathcal{H}_K = \bigvee_{n\in\mathbb{Z}} V(n)\mathcal{H}_n$.

(c) The decomposition is unique in the following sense: if $\mathcal{H}'$ is another Hilbert space and V' defined on $\mathbb{Z}$ is an application such that $V'(n) \in L(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ that satisfy (a) *and* (b)*,then there exists a unitary operator $\Phi : \mathcal{H}_K \to \mathcal{H}'$ such that $\Phi V(n) = V'(n)$ for all $n \in \mathbb{Z}$.*

A proof of this theorem can be found in [3, Teorema 3.1].
An application V that satisfies the property (a) in the former theorem is called *The Kolmogorov Decomposition of the Kernel* K or simply, a *Decomposition of the kernel* K (see [3]). The property (b) is known as *the minimality condition* of Kolmogorov Decomposition. The meaning of property (c) is that, given the minimality condition (b), the Kolmogorov Decomposition is essentially unique.

3 Equivalent Positive Definite Kernels To Operator Values, biequivalent kernels, and dual relation

In this section, we present some of the results given in [1]. In what follows, we will assume that $\mathcal{H}$ is a separable Hilbert space. Inspired by the theory of Riesz bases (see, [11]), we define and study a new class of positive definite kernels. Also deduce some results for positive definite kernels by using the equivalent kernels and Kolmogorov decompositions.

Definition 12. Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to L(\mathcal{H})$ be two positive definite kernels.
We say that K_1 and K_2 are *equivalent* if there exist two constants A, B with $0 < A \le B$ such that

$$A\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}} \le \|[h]_{K_2}\|^2_{\mathcal{H}_{K_2}} \le B\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}},$$

for $h \in \mathcal{F}_o$.

Remark 13. Let $K : \mathbb{Z}\times\mathbb{Z} \to L(\mathcal{H})$ is a positive definite kernel. Let $h \in \mathcal{F}_o$ and $\{h_n\}_{n\in\mathbb{Z}}$ a sequence in $\mathcal{H}$ with finite support.
From the definition of the norm induced by the kernel K and the Kolmogorov decomposition theorem, we have

$$\begin{aligned}\|[h]\|^2_{\mathcal{H}_K} = \langle [h], [h]\rangle_{\mathcal{H}_K} &= \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_K^*(n)V_K(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \left\| \sum_{n\in\mathbb{Z}} V_K(n)h_n \right\|^2_{\mathcal{H}}.\end{aligned}$$

Theorem 14. *Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ are two positive definite kernels. Then the following conditions are equivalent:*

(i) Kernels K_1 and K_2 are equivalent.

(ii) There exists a bijective bounded linear map with bounded inverse,

$$\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$$

such that

$$\Phi V_{K_1}(n) = V_{K_2}(n) \qquad \text{for all } n \in \mathbb{Z}.$$

(iii) There exist are two constants A, B with $0 < A \le B$ such that

$$\begin{aligned}A \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &\le \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\le B \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\end{aligned}$$

for every sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

For the proof of this see [1].

3.1 Equivalent kernels: main results

Extensions for kernels are given to the operator values of the results obtained in [2] for the scalar case. These results are motivated by the importance and applications of dual bases, dual frames, and wavelets and their great relevance in pure and applied mathematics, see [11, 6, 7, 10, 8, 9]. We show that two kernels K_2 and K are equivalent and satisfy a dual relation similar to Riesz bases and biorthogonal sequences.

Proposition 15. *Let $K_1 : \mathbb{Z}\times\mathbb{Z} \to L(\mathcal{H})$ and $K : \mathbb{Z}\times\mathbb{Z} \to L(\mathcal{H})$ are two positive definite kernels such that K_1 and K are equivalent. Then there exists a unique positive definite kernel $K_2 : \mathbb{Z}\times\mathbb{Z} \to L(\mathcal{H})$ with $V_{K_2}(n) \in \mathcal{H}_{K_1}$ for all $n \in \mathbb{Z}$ such that*

$$V^*_{K_1}(n)V_{K_2}(m) = K(n,m) \quad \textit{for every} \quad m, n \in \mathbb{Z}. \tag{3.1}$$

Furthermore, the kernel K_2 is equivalent to K_1. In this case, it is said to be that (K_1, K_2) is a pair of biequivalent positive definite kernels with kernel K.

Proof. Since K_1 and K are equivalent, by Theorem 14, there exists a bounded invertible operator $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_K$ such that

$$\Phi V_{K_1}(n) = V_K(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Let $K_2 : \mathbb{Z} \times \mathbb{Z} \to L(\mathcal{H})$ the kernel given by

$$K_2(n,m) = (\Phi^*\Phi V_{K_1}(n))^*\Phi^*\Phi V_{K_1}(m)$$

for every $m, n \in \mathbb{Z}$.
It is easily seen that K_2 is a positive definite kernel. For any sequence with finite support $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$, it follows that

$$\begin{aligned}
&\sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} = ...\\
&... = \sum_{n,m\in\mathbb{Z}} \langle (\Phi^*\Phi V_{K_1}(n))^*\Phi^*\Phi V_{K_1}(m)h_m, h_n\rangle_{\mathcal{H}}\\
&= \sum_{n,m\in\mathbb{Z}} \langle \Phi^*\Phi V_{K_1}(m)h_m, \Phi^*\Phi V_{K_1}(n)h_n\rangle_{\mathcal{H}_{K_1}}\\
&= \left\langle \sum_{m\in\mathbb{Z}} \Phi^*\Phi V_{K_1}(m)h_m, \sum_{n\in\mathbb{Z}} \Phi^*\Phi V_{K_1}(n)h_n \right\rangle_{\mathcal{H}_{K_1}}\\
&= \left\| \sum_{m\in\mathbb{Z}} \Phi^*\Phi V_{K_1}(m)h_m \right\|^2_{\mathcal{H}_{K_1}} \geq 0.
\end{aligned}$$

Then $V_{K_2}(n) \in \mathcal{H}_{K_1}$ for all $n \in \mathbb{Z}$ and

$$\begin{aligned}
V^*_{K_1}(n)V_{K_2}(m) &= V^*_{K_1}(n)\Phi^*\Phi V_{K_1}(m)(\Phi V_{K_1}(n))^*\Phi V_{K_1}(m)\\
&= V^*_K(n)V_K(m) = K(n,m).
\end{aligned}$$

for every $m, n \in \mathbb{Z}$.

Suppose now that K_3 is a positive definite kernel such that $V_{K_3}(m) \in \mathcal{H}_{K_1}$ which also satisfies 3.1. Then for any fixed m

$$V^*_{K_1}(n)V_{K_2}(m) = V^*_{K_1}(n)V_{K_3}(m) \qquad \text{for all} \qquad n \in \mathbb{Z},$$

so we must have $V_{K_2}(m) - V_{K_3}(m) = 0$. This shows that K_2 is unique. Finally, since Φ is bounded and invertible, the adjoint operator Φ^* is also bounded and invertible and $\Phi^*V_K(m) = V_{K_2}(m)$ for each $m \in \mathbb{Z}$, then K_2 and K are equivalent. Now, being the kernels K and K_1 equivalent, we have by the transitivity property of the equivalence of kernels that K_2 is equivalent to K_1. □

In analogy with Proposition 9 (see also [4, 6, 8, 9]) we have the following.

Proposition 16 (dual relation)**.** *Let $K_1 : \mathbb{Z} \times \mathbb{Z} \to L(\mathcal{H})$ and $K : \mathbb{Z} \times \mathbb{Z} \to L(\mathcal{H})$ have two definite positive kernels such that K_1 and K are equivalent, and let $K_2 : \mathbb{Z} \times \mathbb{Z} \to L(\mathcal{H})$ with $V_{K_2}(n) \in \mathcal{H}_{K_1}$ for all $n \in \mathbb{Z}$ which satisfy the following relation*

$$V^*_{K_1}(n)V_{K_2}(m) = K(n,m) \quad \text{for every} \quad m, n \in \mathbb{Z}.$$

Then, there exist positive constants $A \leq B$ such that

$$\begin{aligned}
A \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\\
&\leq B \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}
\end{aligned}$$

Furthermore, one has the dual relation

$$\begin{aligned}
\frac{1}{B} \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}}\\
&\leq \frac{1}{A} \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}
\end{aligned}$$

for every sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

Proof. The assumption that K_1 and K are equivalent implies there exist positive constants $A \leq B$ such that

$$\begin{aligned}
A \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\\
&\leq B \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}.
\end{aligned}$$

Now show that one has the dual relation. Since K_1 and K are equivalent, by Theorem 14, there exists a bounded invertible operator $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_K$ such that

$$\Phi V_{K_1}(n) = V_K(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

As Φ is bounded and invertible, the adjoint operator Φ^* is also bounded and invertible.
Assertion

$$\Phi^*\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

Indeed, we have

$$\begin{aligned}
V^*_{K_1}(n)V_{K_2}(m) &= V^*_{K_1}(n)\Phi^*\Phi V_{K_1}(m) = (\Phi V_{K_1}(n))^*\Phi V_{K_1}(m)\\
&= V^*_K(n)V_K(m) = K(n,m).
\end{aligned}$$

for every $m, n \in \mathbb{Z}$.

Let $f \in \mathcal{H}_K$ given by

$f = \sum_{n\in\mathbb{Z}} V_K(n)h_n$, where $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$ with finite support.

Then

$$\begin{aligned}\left\langle \left(\Phi\Phi^*\right)^{-1} f, f\right\rangle_{\mathcal{H}_K} &= \left\langle \left(\Phi^{-1}\right)^* \Phi^{-1} f, f\right\rangle_{\mathcal{H}_K} \\ &= \left\langle \Phi^{-1} f, \Phi^{-1} f\right\rangle_{\mathcal{H}_{K_1}} \\ &= \left\|\Phi^{-1} f\right\|^2_{\mathcal{H}_{K_1}} \\ &= \left\|\Phi^{-1}\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|^2_{\mathcal{H}_{K_1}} \\ &= \left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}} \\ &= \sum_{n,m\in\mathbb{Z}} \left\langle K_1(n,m)h_m, h_n\right\rangle_{\mathcal{H}}.\end{aligned}$$

Considering this and the inequality (2.1), of Proposition 6 we have that

$$A\left\|f\right\|^2_{\mathcal{H}_K} \le \left\langle \left(\Phi\Phi^*\right)^{-1} f, f\right\rangle_{\mathcal{H}_K} \le B\left\|f\right\|^2_{\mathcal{H}_K}.$$

Since $f \in \mathcal{H}_K$ is arbitrary, one gets that the linear operator $\left(\Phi\Phi^*\right)^{-1}$ is positive, then by Proposition 6, we have

$$\frac{1}{B}\left\|f\right\|^2_{\mathcal{H}_K} \le \left\langle \Phi\Phi^* f, f\right\rangle_{\mathcal{H}_K} \le \frac{1}{A}\left\|f\right\|^2_{\mathcal{H}_K}.$$

Given that,

$$\left\langle \Phi\Phi^* f, f\right\rangle_{\mathcal{H}_K} = \left\langle \Phi^* f, \Phi^* f\right\rangle_{\mathcal{H}_{K_2}} = \left\|\Phi^* f\right\|^2_{\mathcal{H}_{K_2}}$$
$$= \left\|\sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n\right\|^2_{\mathcal{H}_{K_2}} = \sum_{n,m\in\mathbb{Z}} \left\langle K_2(n,m)h_m, h_n\right\rangle_{\mathcal{H}},$$

□

concludes the proof.

4 Multivariate Stochastic Processes, and equivalent Multivariate Stochastic Processes

In this section it will be used for the decomposition of the covariance Kernels between the stochastic processes (see, [3], section 1, Chapter 6). We obtain new results for stochastic processes.

4.1 Multivariate Stochastic Processes

Definition 17. A pair $[\mathcal{K}, X]$, where $\mathcal{K}$ is a Hilbert space and $X = \{X_n\}_{n\in\mathbb{Z}}$ is a family of operators in $\mathcal{L}(\mathcal{H}_n, \mathcal{K})$, is called a *geometric model of the multivariate process* with covariance kernel K, if

$$K(m,n) = X_m^* X_n.$$

The Kolmogorov Decomposition Theorem shows that given a positive definite kernel K, there exists a geometric model of the multivariate process with covariance kernel K. If $[\mathcal{K}, X]$ is the geometric model of the multivariate process with covariance kernel K then $\mathcal{H}_X$ will be the subspace of $\mathcal{K}$ generated by this model, that is,

$$\mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n\mathcal{H}_n. \tag{4.1}$$

If $[\mathcal{K}', X']$ is another geometric model of the same process, then the Kolmogorov Decomposition Theorem guarantees the existence of a unitary operator $\Phi : \mathcal{H}_X \to \mathcal{H}_{X'}$ such that $\Phi X_n = X'_n$ for all $n \in \mathbb{Z}$. This means that the geometry of the process is essentially determined by the choice of a geometric model such that

$$\mathcal{K} = \bigvee_{n\in\mathbb{Z}} X_n\mathcal{H}_n. \tag{4.2}$$

4.2 Equivalent Multivariate Stochastic Processes

From here on, $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$ and the covariance kernel of the processes will be positive definite.

Theorem 18 (Isomorphism)**.** *Let $[\mathcal{W}, X]$ is a geometric model of the multivariate process and $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be the covariance kernel of the processes, then there exists a unitary operator $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ such that*

$$\Phi V_K(n) = X_n \qquad \textit{for every} \qquad n \in \mathbb{Z}.$$

Proof. Let $[\mathcal{W}, X]$, $X = \{X_n\}_{n\in\mathbb{Z}}$ is a geometric model of a multivariate process and $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ is the kernel of covariance associated with the process.
It follows that the covariance kernel and the space generated by the process are given by

$$K(n,m) = X_n^* X_m \quad \text{and} \quad \mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n\mathcal{H}.$$

On the other hand, since K is a positive definite kernel, one more time by the Kolmogorov decomposition theorem, there exists a Hilbert space $\mathcal{H}_K$ and an application $V_K(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$ such that

$$K(n,m) = V_K^*(n)V_K(m) \quad \text{and} \quad \mathcal{H}_K = \bigvee_{n\in\mathbb{Z}} V_K(n)\mathcal{H}.$$

Let us define the application $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ in the following way

$$\Phi\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right) = \sum_{n\in\mathbb{Z}} X_n h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence with finite support in $\mathcal{H}$.

Then we have

$$\begin{aligned}\left\|\Phi\left(\sum_{n\in\mathbb{Z}}V_K(n)h_n\right)\right\|^2_{\mathcal{H}_X} &= \left\|\sum_{n\in\mathbb{Z}}X_nh_n\right\|^2_{\mathcal{H}_X}\\ &= \sum_{m,n\in\mathbb{Z}}\langle X_mh_m, X_nh_n\rangle_{\mathcal{H}_X}\\ &= \sum_{m,n\in\mathbb{Z}}\langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}\\ &= \sum_{m,n\in\mathbb{Z}}\langle V_K^*(n)V_K(m)h_m, h_n\rangle_{\mathcal{H}}\\ &= \sum_{m,n\in\mathbb{Z}}\langle V_K(m)h_m, V_K(n)h_n\rangle_{K}\\ &= \left\|\sum_{n\in\mathbb{Z}}V_K(n)h_n\right\|^2_{\mathcal{H}_K}.\end{aligned}$$

All of this show us that the application Φ can be extended by continuity to an unit operator from $\mathcal{H}_K$ over $\mathcal{H}_X$ and moreover $\Phi V_K(n) = X_n$ for all $n\in\mathbb{Z}$. □

Definition 19. Two geometric models of multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are said to be *equivalent* , if $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$ and there are two constants A, B with $0 < A \le B$ such that

$$A\left\|\sum_{n\in\mathbb{Z}}X_nh_n\right\|^2_{\mathcal{H}_X} \le \left\|\sum_{n\in\mathbb{Z}}Y_nh_n\right\|^2_{\mathcal{H}_Y} \le B\left\|\sum_{n\in\mathbb{Z}}X_nh_n\right\|^2_{\mathcal{H}_X},$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.

From the Isomorphism Theorem 18 and the definitions, we have the following.

Proposition 20. *Let $[\mathcal{W}, X]$ and $[\mathcal{W}_1, Y]$ be two geometric models of multivariate stochastic processes and K_1, K_2 the covariance kernels associated with the processes respectively. Then K_1 and K_2 are equivalent kernels if and only if $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent processes.*

Theorem 21. *Let $[\mathcal{K}, X]$ y $[\mathcal{L}, Y]$ two geometric models of multivariate processes. The following conditions are equivalent:*

(i) The models of the multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are equivalent.

(ii) There is a bijective bounded linear application with bounded inverse

$$\psi : \mathcal{H}_X \to \mathcal{H}_Y$$

such that

$$\psi X_n = Y_n \qquad \textit{for all } n\in\mathbb{Z}.$$

(iii) There exist two constants A, B with $0 < A \le B$ such that

$$\begin{aligned}A\sum_{n,m\in\mathbb{Z}}\langle X_n^*X_mh_m, h_n\rangle_{\mathcal{H}} &\le \sum_{n,m\in\mathbb{Z}}\langle Y_n^*Y_mh_m, h_n\rangle_{\mathcal{H}}\\ &\le B\sum_{n,m\in\mathbb{Z}}\langle X_n^*X_mh_m, h_n\rangle_{\mathcal{H}},\end{aligned}$$

for each sequence with finite support $\{h_n\}_{n\in\mathbb{Z}}\subset\mathcal{H}$.

A proof of this theorem can be found in [1].

Proposition 22. *Let $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ two geometric models of multivariate processes such that $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are equivalents, and let $[\mathcal{M}, Z]$ be a geometric model of a multivariate process such that $Z = \{Z_n\}_{n\in\mathbb{Z}}$ with $Z_n\in\mathcal{H}_X$ for all $n\in\mathbb{Z}$ which satisfy the following relation*

$$X_n^*Z_m = Y_n^*Y_m \quad \textit{for all} \quad m,n\in\mathbb{Z}.$$

Then, there exist positive constants $A \le B$ such that

$$A\left\|\sum_{n\in\mathbb{Z}}Y_nh_n\right\|^2_{\mathcal{H}_Y} \le \left\|\sum_{n\in\mathbb{Z}}X_nh_n\right\|^2_{\mathcal{H}_X} \le B\left\|\sum_{n\in\mathbb{Z}}Y_nh_n\right\|^2_{\mathcal{H}_Y} \tag{4.3}$$

Furthermore, one has the following dual relation

$$\frac{1}{B}\left\|\sum_{n\in\mathbb{Z}}Y_nh_n\right\|^2_{\mathcal{H}_Y} \le \left\|\sum_{n\in\mathbb{Z}}Z_nh_n\right\|^2_{\mathcal{H}_X} \le \frac{1}{A}\left\|\sum_{n\in\mathbb{Z}}Y_nh_n\right\|^2_{\mathcal{H}_Y} \tag{4.4}$$

for each sequence with finite support $\{h_n\}_{n\in\mathbb{Z}}\subset\mathcal{H}$.

Proof. Let $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ the geometric models of multivariate processes and K_1 and K are the covariance kernel associated with the stochastic processes, respectively. Since $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are equivalent , we have that the kernels K_1 and K are equivalent. Then there exists a unique positive definite kernel $K_2 : \mathbb{Z}\times\mathbb{Z}\to L(\mathcal{H})$ with $V_{K_2}(n)\in\mathcal{H}_{K_1}$ for all $n\in\mathbb{Z}$ such that

$$V_{K_1}^*(n)V_{K_2}(m) = K(n,m) \quad \text{for every} \quad m,n\in\mathbb{Z}.$$

Now, by Proposition 16, there exist positive constants $A\le B$ such that

$$\begin{aligned}A\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\le \sum_{n,m\in\mathbb{Z}}\langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\\ &\le B\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}\end{aligned}$$

Furthermore, one has the dual relation

$$\begin{aligned}\frac{1}{B}\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\le \sum_{n,m\in\mathbb{Z}}\langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}}\\ &\le \frac{1}{A}\sum_{n,m\in\mathbb{Z}}\langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}\end{aligned}$$

for every sequence with finite support $\{h_n\}_{n\in\mathbb{Z}}\subset\mathcal{H}$. Therefore, if

$$K_2(n,m) = Z_n^*Z_m \quad \text{for all} \quad m,n\in\mathbb{Z}.$$

Then the statement holds since

$$\left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|^2_{\mathcal{H}_Y} = \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}},$$

$$\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} = \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}$$

and

$$\left\|\sum_{n\in\mathbb{Z}} Z_n h_n\right\|^2_{\mathcal{H}_X} = \sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}}.$$

□

5 Conclusions

In this paper, the existence of a positive definite kernel is guaranteed for every pair of equivalent positive definite kernels. We have used Kolmogorov decomposition theorem and equivalent kernels. Furthemore, we show a dual relation in the context of positive definite kernels of results concerning Riesz bases and dual frames in Hilbert spaces as well as in Krein spaces, see [11, 4, 6, 8]. Also we obtain new results for stochastic processes. In future work we could think give new results and establishing connections with the theory of frames , Riesz bases and nontrivial examples of this research in generalized Hilbert spaces.

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