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Connected, Compactness and separation axioms via (i,j)-α^m-open sets in bitopological spaces

Conexidad, compacidad y axiomas de separación vía conjuntos (i,j)-α^m-abiertos en espacios bitopológicos

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Resumen

En este artículo, se utiliza la noción de conjuntos (i,j)-α^m-abiertos para introducir las nociones de (i,j)-α^m-conexo, (i,j)-α^m-compacto, (i,j)-α^m-T_0-espacio, (i,j)-α^m-T_1-espacio e (i,j)-α^m-T_2-espacio. Adicionalmente, algunas de sus propiedades y caracterizaciones son probadas.

Palabras clave: Espacios bitopológicos, generalizaciones de conjuntos abiertos, conexidad y compacidad .

Abstract:

In this paper, we use the notion of (i,j)-α^m-open sets to introduce the concepts of (i,j)-α^m-connected, (i,j)-α^m-compact, (i,j)-α^m-T_0-space, (i,j)-α^m-T_1-space and (i,j)-α^m-T_2-space. Furthermore, we prove and show some of their properties and characterizations.

Keywords: Bitopological spaces, generalized open sets, connected and compactness.

1. Introduction and preliminaries.

The notion of bitopological space was introduced by Kelly [8] in 1963, the study of open and closed sets in bitopological spaces have increased in several field of general topology. Additionally, it is well know that the concept of compactness is one of the most important subject in general topology and it has a very important role in the theory of topological spaces, bitopological spaces and much more. On the other hand, connected spaces is very important in general topology. Recently, Granados [5] studied ω-$\mathcal{N}$-α-open sets on connected spaces and showed some of their properties. Furthermore, the notion of (i,j)-α^m-open sets, (i,j)-α-continuous and conta (i,j)-α^m-continuous functions were introduced by Granados see([4, 3]).

In this paper, we show and investigate various properties of (i,j)-α^m-compact, (i,j)-α^m-connected, (i,j)-α^m-T_0-space, (i,j)-α^m-T_1-space, (i,j)-α^m-T_2-space. Besides, we prove some of their properties.

Throughout this paper, (X,τ_1,τ_2) means a bitopological space on which no separation axioms are assumed unless otherwise mentioned. Moreover, we sometimes write X instead of (X,τ_1,τ_2).

Now, we write some definitions which are useful for the developing of this paper.

Definition 1. *Let (X,τ_1,τ_2) be a bitopological space and $A\subset X$, then A is said to be (i,j)-α^m-closed set [4] if $Int_{\tau_i}(Cl_{\tau_j}(A))\subseteq U$, whenever $A\subseteq U$ and U is (i,j)-α-open set where $(i,j)\in\{1,2\}$. The complement of an (i,j)-α^m-closed set is called (i,j)-α^m-open set.*

Example 1. *Let $X=\{q,w,e,r\}$, $\tau_i=\{\emptyset,X,\{q\},\{w\},\{q,w\},\{q,w,r\}\}$ and $\tau_j=\{\emptyset,X,\{q\},\{e\},\{q,e\},\{q,e,r\}\}$. Then, $\{q,w\}$ is an (i,j)-α^m-closed set.*

Definition 2. *[4] The intersection of all (i,j)-α^m-closed sets of X containing A is called the (i,j)-α^m-closure of A and it is denoted by (i,j)-α^m-$Cl(A)$.*

Definition 3. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-α^m-continuous [4] if $f^{-1}(V)$ is (i,j)-α^m-closed (respectively, (i,j)-α^m-open) set of X for every σ_j-closed (respectively, σ_j-open) set V of Y where $(i,j)\in\{1,2\}$.*

Definition 4. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-α^m-irresolute if [4] $f^{-1}(V)$ is (i,j)-α^m-closed (re-*

spectively, (i,j)-α^m-open) set of X for every (i,j)-α^m-closed (respectively, (i,j)-α^m-open) set V of Y where $(i,j) \in \{1,2\}$.

Definition 5. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be contra (i,j)-α^m-continuous [3] if $f^{-1}(V)$ is (i,j)-α^m-closed set of X for every σ_j-open set V of Y where $(i,j) \in \{1,2\}$.*

Definition 6. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be contra (i,j)-α^m-irresolute if [3] $f^{-1}(V)$ is (i,j)-α^m-closed set of X for every (i,j)-α^m-open set V of Y where $(i,j) \in \{1,2\}$.*

Definition 7. *A collection U of a bitopological space (X,τ_1,τ_2) is said to be pairwise open [2] if $U \subset \tau_i \cup \tau_j$ and for each $(i,j) \in \{1,2\}$, $U \cap \tau_i$ contains a non-empty set. Besides, U is said to be pairwise open cover of X if U covers X.*

Definition 8. *A bitopological space (X,τ_1,τ_2) is said to be pairwise compact [2] if every pairwise open cover of X has a finite subcover.*

Definition 9. *Let (X,τ_1,τ_2) be a bitopolgical space. Then, X si said to be pairwise C-compact [9] if for each pair of point τ_j-closed A of X and each subsets pairwise open on $\mathcal{U}$ of A, exits a finite sub-family of elements $\mathcal{U}$, $V_1, V_2, V_3, ..., V_n$ such that $A \subset \bigcup_{i=1}^{n} \tau_j\text{-}ClV_j$, where $i \neq j$ where $(i,j) \in \{1,2\}$.*

2. Separation axioms via (i,j)-α^m-open sets

In this section, we define and study the concepts of (i,j)-α^m-T_0-space, (i,j)-α^m-T_1-space and (i,j)-α^m-T_2-space. Throughout this section (X,τ_1,τ_2) is a bitopological space where $(i,j) \in \{1,2\}$.

Definition 10. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-α^m-T_0 if for every pair of distinct points in X, there exits an (i,j)-α^m-open set of X containing one of points but not the other.*

Theorem 1. *A bitopological space (X,τ_1,τ_2) is (i,j)-α^m-T_0 if and only if for each pair of distinct points x, y of X, (i,j)-α^m-$Cl(\{x\}) \neq (i,j)$-α^m-$Cl(\{y\})$*

Proof. Let (X,τ_1,τ_2) be an (i,j)-α^m-T_1 space and x, y any two distinct points of X. Then, there exits an (i,j)-α^m-open set V containing x or y, say, x but not y. Then, $X - V$ is an (i,j)-α^m-closed set, that does not contain x but contains y. Since (i,j)-α^m-$Cl\{y\})$ is the smallest (i,j)-α^m-closed set containing y, (i,j)-α^m-$Cl(\{y\}) \subset X - V$, and so $x \notin (i,j)$-α^m-$Cl(\{y\})$. Consequently, (i,j)-α^m-$Cl(\{x\}) \neq (i,j)$-α^m-$Cl(\{y\})$. Conversely, let $x, y \in X$, $x \neq y$ and (i,j)-α^m-$Cl(\{x\}) \neq (i,j)$-α^m-$Cl(\{y\})$. Then, there exits a point $z \in X$ such that z belongs to one of the two sets, say, (i,j)-α^m-$Cl(\{x\})$ but no to (i,j)-α^m-$Cl(\{y\})$. Suppose that $x \in (i,j)$-α^m-$Cl(\{y\})$, then $z \in (i,j)$-α^m-$Cl(\{x\}) \subset (i,j)$-α^m-$Cl(\{y\})$, which is a contradiction with $z \notin (i,j)$-α^m-$Cl(\{y\})$. Therefore, $x \in X - (i,j)$-α^m-$Cl(\{y\})$, but $X - (i,j)$-α^m-$Cl(\{y\})$ is an (i,j)-α^m-open set and does not contain y. In consequence, (X,τ_1,τ_2) is (i,j)-α^m-T_0. □

Definition 11. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-α^m-T_1 if fore every pair of distinct points x, y of X, there exits a pair (i,j)-α^m-open sets one containing x but not y and the other containing y but not x.*

Remark 1. *It is clear that every (i,j)-α^m-T_1 is (i,j)-α^m-T_0, but the converse need not be true.*

Theorem 2. *For a bitopological space (X,τ_1,τ_2), the following statements are equivalent:*

1. *(X,τ_1,τ_2) is (i,j)-α^m-T_1.*
2. *Each singleton subset of X is (i,j)-α^m-closed in X.*
3. *Each subset of X is the intersection of all (i,j)-α^m-open sets containing it.*
4. *The intersection of all (i,j)-α^m-open sets containing the point $x \in X$ is the set $\{x\}$.*

Proof. (1)⇒(2): Let $x \in X$. Then, by the part (1) of this Theorem, for any $y \in X$, $y \neq x$, there exits an (i,j)-α^m-open set U_y containing y but not x. Indeed, $y \in U_y \subset X - \{x\}$. Now, varying y over $X - \{x\}$, we have $X - \{x\} = \bigcup\{U_y : y \in X - \{x\}\}$. Hence, $X - \{x\}$ being an union of (i,j)-α^m-open sets. Therefore, $\{x\}$ is (i,j)-α^m-closed.

(2)⇒(3): If $U \subset X$, then for each point $y \notin U$, there exits a set $X - \{y\}$ such that $U \subset X - \{y\}$ and each of these sets $X - \{y\{$ is (i,j)-α^m-open. Therefore, $U = \bigcap\{X - \{y\} : y \in X - U\}$ and so the intersection of all (i,j)-α^m-open sets containing U is the set U itself.

(3)⇒(4): It is clear.

(4)⇒(1): Let $x, y \in X$ and $x \neq y$. Then, there exits an (i,j)-α^m-open sets, say, V_x such that $y \notin V_x$. Similarly, there exits an (i,j)-α^m-open set $V_y, x \notin V_y$. Therefore, (X,τ_1,τ_2) is (i,j)-α^m-T_1. □

Lemma 1. *If every finite subset of a bitopological space (X,τ_1,τ_2) is (i,j)-α^m-closed, then it is (i,j)-α^m-T_1.*

Proof. Let $x, y \in X$ such that $x \neq y$. Then, by hypothesis, $\{x\}$ and $\{y\}$ are (i,j)-α^m-closed sets in X. Thus, $X - \{x\}$ and $X - \{y\}$ are (i,j)-α^m-open sets of X such that $x \in X - \{y\}$ and $y \in X - \{x\}$. Therefore, (X,τ_1,τ_2) is (i,j)-α^m-T_1. □

Definition 12. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-α^m-T_2 if for every pair of distinct points x, y of X, there exits a pair of disjoint (i,j)-α^m-open sets, one containing x and the other containing y.*

Remark 2. *It is clear that every (i,j)-α^m-T_2 is (i,j)-α^m-T_1, but the converse need not be true.*

Theorem 3. *For a bitopological space (X,τ_1,τ_2), the following statements are equivalent:*

1. *(X,τ_1,τ_2) is (i,j)-α^m-T_2.*

2. *Let $x \in X$. For each $y \neq x$, there exits $U \in (i,j)$-$\alpha^m BO(X,x)$ and $y \notin (i,j)$-α^m-$Cl(U)$.*

3. *For ech $x \in X$, $\bigcap\{(i,j)$-α^m-$Cl(V_x) : V_x$ is an (i,j)-α^m-open set containing x $\} = \{x\}$.*

Proof. (1)⇒(2): Let $x \in X$ and $y \neq x$. Then, there exits disjoint (i,j)-α^m-open sets U and V such that $x \in U$ and $y \in V$. Clearly, $X - V$ is (i,j)-α^m-closed. Indeed, (i,j)-α^m-$Cl(U) \subset X - V$ and then $y \notin (i,j)$-α^m-$Cl(U)$.

(2)⇒(3): If $y \neq x$, then there exits $V \in (i,j)$-$\alpha^m BO(X,x)$ and $y \notin (i,j)$-α^m-$Cl(V)$. Therefore, $y \notin \bigcap\{(i,j)$-α^m-$Cl(V) : V \in (i,j)$-$\alpha^m BO(X,x)\}$.

(3)⇒(1): Let $x, y \in X$ such that $x \neq y$. Then, $y \notin \{x\} = \bigcap\{(i,j)$-$\alpha^m$-$Cl(V) : V \in (i,j)$-$\alpha^m BO(X,x)\}$. Therefore, $y \notin (i,j)$-α^m-$Cl(V)$ for some (i,j)-α^m-open set V containing x. Clearly, V and $X - (i,j)$-α^m-$Cl(V)$ are the required (i,j)-α^m-open sets containing x and y, respectively. □

Definition 13. *If A is both (i,j)-α^m-open set and (i,j)-α^m-closed set of a bitopological space (X,τ_1,τ_2), then A is called (i,j)-α^m-coplen, where $i \neq j$.*

Theorem 4. *A bitopological space (X,τ_1,τ_2) is (i,j)-α^m-T_2 if and only if for each pair of distinct points $x, y \in X$, there exits an (i,j)-α^m-coplen set V containing one of them but not the other.*

Proof. Let (X,τ_1,τ_2) be an (i,j)-α^m-T_2 space and $x, y \in X$ such that $x \neq y$ implies that there exits two disjoint (i,j)-α^m-open sets U and V such that $x \in U$ and $y \in V$. Since $U \cap V = \emptyset$ and V is an (i,j)-α^m-open set, $x \in U \subset X - V$ and $X - V$ is (i,j)-α^m-closed. Since (X,τ_1,τ_2) is (i,j)-α^m-T_2 for each $x \in X - V$ there exits and (i,j)-α^m-open set U_x such that $x \in U_x \subset X - V$. Then, $X - V$ is (i,j)-α^m-open. In consequence, $X - V$ is an (i,j)-α^m-coplen set. Conversely, suppose for every pair of distinct points $x, y \in X$, there exits an (i,j)-α^m-coplen set U containing x but not y. Then, $X - U$ is an (i,j)-α^m-open set and $y \in X - U$. Since $U \cap (X - U) = \emptyset$, (X,τ_1,τ_2) is an (i,j)-α^m-T_2 space. □

Theorem 5. *For a bitopological space (X,τ_1,τ_2), the following statements are equivalent:*

1. *(X,τ_1,τ_2) is (i,j)-α^m-T_2.*

2. *The intersection of all (i,j)-α^m-coplen sets of each point in X is singleton.*

3. *For a finite number of distinct points $\{x_i : 1 \leq i \leq n\}$, there exits an (i,j)-α^m-open set H_i such that $\{H_i : 1 \leq i \leq n\}$ are pairwise disjoint.*

Proof. (1)⇒(2): Let (X,τ_1,τ_2) be an (i,j)-α^m-T_2 space and $x \in X$. Suppose $\bigcap\{H : H$ is (i,j)-α^m-coplen and $x \in H\} = \{x, y\}$ where $x \neq y$. Since (X,τ_1,τ_2) is (i,j)-α^m-T_2, there exists two disjoint (i,j)-α^m-open sets U and V such that $x \in U$ and $y \in V$. Then, $x \in U \subset X - V$, indeed $X - V$ is (i,j)-α^m-open set and also it is (i,j)-α^m-closed. Therefore, $X - V$ is (i,j)-α^m-coplen set containing x but not y, which is a contradiction. This implies that the intersection of all (i,j)-α^m-coplen sets containing x is $\{x\}$.

(2)⇒(3): Let $\{x_1, x_2, ..., x_n\}$ be a finite number of distinct points of X. Then, by part (2) of this Theorem, $\{x_i = \bigcap\{B : B$ is (i,j)-α^m-coplen set and $x_i \in B\}$ for $i = 1, 2, ..., n$. Since $x_j \in \{x_i\}$, for $i, j = 1, 2, ..., n$ and $i \neq j$, there exits an (i,j)-α^m-coplen set B_0 such that $x_i \in B_0$ and $x_j \notin B_0$ for $i \neq j$ and $1 \leq i, j \leq n$. Then, $x_i \in X - B_0$, where $X - B_0$ is an (i,j)-α^m-coplen set and $B_0 \cap (X - B_0) = \emptyset$. Indeed, $X - B_0$ is an (i,j)-α^m-open set containing x_i. Therefore, for each i, there exits pairwise disjoint (i,j)-α^m-open sets H_i for $\{x_i : 1 \leq i \leq n\}$.

(3)⇒(1): It is clear. □

3. (i,j)-α^m-connected

In this section, we define and study the concepts of (i,j)-α^m-connected space, (i,j)-α^m-set-connected and (i,j)-α^m-extremally disconnected. Besides we show some results on (i,j)-α^m-continuous functions. Throughout this section (X,τ_1,τ_2) is a bitopological space where $(i,j) \in \{1,2\}$.

Definition 14. *Let (X,τ_1,τ_2) be a bitopological space. Then, X is said to be (i,j)-α^m-connected if X cannot be expressed as the union of two non-empty disjoint (i,j)-α^m-open sets.*

Definition 15. *Let (X,τ_1,τ_2) be a bitopological space. Then, X is said to be pairwise connected [10] if it cannot be expressed as the union of two non-empty disjoint sets U and V such that U is τ_i-open and V is τ_j-open, where $i \neq j$.*

Proposition 1. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a (i,j)-α^m-continuous and surjection function, besides X is (i,j)-α^m-connected, then Y is pairwise connected.*

Proof. Suppose that Y is not pairwise connected. Then, there exits $U \in \sigma_i$ and $V \in \sigma_j$ such that $U, V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = Y$. Since f is surjection, it has $f^{-1}(U) \neq \emptyset$ and $f^{-1}(V) \neq \emptyset$. Besides, since f is (i,j)-α^m-continuous, $f^{-1}(U)$ is (i,j)-α^m-continuous, it has $f^{-1}(U)$ is (i,j)-α^m-open and $f^{-1}(V)$ is (i,j)-α^m-open such that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. This implies that X is not (i,j)-α^m-connected, which is a contradiction. In consequence Y is pairwise connected. □

Proposition 2. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is a (i,j)-α^m-irresolute and surjection function, besides X is (i,j)-α^m-connected, then Y is (i,j)-α^m-connected.*

Proof. The proof is similar to the Proposition 1. □

Definition 16. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-α^m-set-connected if $f(x)$ is (i,j)-α^m-connected between $f(A)$ and $f(B)$ in the bitopological space X which is (i,j)-α^m-connected.*

Theorem 6. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function (i,j)-α^m-set-connected if and only if $f^{-1}(F)$ is an (i,j)-α^m-coplen set of X for any (i,j)-α^m-coplen F set of Y.*

Proof. **NECESSARY**: Let f be (i,j)-α^m-set-connected and F be (i,j)-α^m-coplen set of Y. Now, suppose that $f^{-1}(F)$ is not (i,j)-α^m-coplen set of X, then X is (i,j)-α^m-connected between $f^{-1}(F)$ and $X-f^{-1}(F)$. Since f is (i,j)-α^m-set-connected, Y is (i,j)-α^m-connected between $f(f^{-1}(F))$ and $f(X-f^{-1}(F))$. But, $f(f^{-1}(F))=F\cap Y=F$ and $f(X-f^{-1}(F))=Y-F$, in consequence F is not (i,j)-α^m-clopen set of Y and this is a contradiction. Therefore, $f^{-1}(F)$ is an (i,j)-α^m-coplen set of X.

SUFFICIENCY: Let $f^{-1}(F)$ be an (i,j)-α^m-clopen set of X for any (i,j)-α^m-coplen F set of Y and let X be (i,j)-α^m-connected between A and B. Now, suppose that Y is not (i,j)-α^m-connected between $f(A)$ and $f(B)$, then there exits an (i,j)-α^m-coplen F set of Y such that $f(A)\subset F\subset Y-f(B)$. But, $A\subset f^{-1}(F)\subset X-B$ and $f^{-1}(F)$ is an (i,j)-α^m-coplen set of X and this is a contradiction, because X is (i,j)-α^m-connected. Therefore, f is (i,j)-α^m-connected. □

Lemma 2. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function (i,j)-α^m-set-connected and $A\subset X$ such that $f(A)$ is an (i,j)-coplen set of Y. Then, the restriction $f|_A:A\to Y$ is (i,j)-α^m-set-connected.*

Proof. Let A be (i,j)-α^m-connected space between B and C. Then, X is (i,j)-α^m-connected between B and C of Y is (i,j)-α^m-connected between $f(B)$ and $f(C)$. Since $f(A)$ is an (i,j)-coplen set of Y, then $f(A)$ is an (i,j)-α^m-connected between $f(B)$ and $f(C)$. □

Definition 17. *Let (X,τ_1,τ_2) be a bitopological space. Then, X is said to be (i,j)-α^m-extremally disconnected if the (i,j)-α^m-closure of any (i,j)-α^m-open set is (i,j)-α^m-open set, where $i\neq j$.*

Theorem 7. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function (i,j)-α^m-set-connected. If Y is (i,j)-α^m-T_2 space and (i,j)-α^m-extremally disconnected, then $f|_A:A\to Y$ is constant for every (i,j)-α^m-connected subset A of X.*

Proof. Let $x,y\in A$ and $x\neq y$. Suppose that $f(x)\neq f(y)$ in Y. Since Y is (i,j)-α^m-T_2 space and (i,j)-α^m-extremally disconnected, there exits (i,j)-α^m-coplen set U of Y such that $f(x)\in U$ and $f(y)\notin U$. Now, since f is (i,j)-α^m-set-connected, it has $f^{-1}(U)$ is (i,j)-α^m-coplen set of X. And so, $f^{-1}(U)\cap A$ is a non-empty proper (i,j)-α^m-coplen set of the subset A, this implies that A is not (i,j)-α^m-connected space and this is a contradiction. Therefore, $f(x)=f(y)$ and hence $f|_A:A\to Y$ is constant. □

Theorem 8. *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a surjective function, then the following properties hold:*

1. *If f is contra (i,j)-α^m-irresolute and (X,τ_1,τ_2) is an (i,j)-α^m-connected space, then (Y,σ_1,σ_2) is a (i,j)-α^m-connected space.*
2. *If f is contra (i,j)-α^m-continuous and (X,τ_i,τ_j) is an (i,j)-α^m-connected space, then (Y,σ_1,σ_2) is a (i,j)-α^m-connected space.*

Proof. (1) Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective contra (i,j)-α^m-irresolute function and (X,τ_1,τ_2) an (i,j)-α^m-connected space. Suppose that (Y,σ_1,σ_2) is not (i,j)-α^m-connected. Then, there exist nonempty (i,j)-α^m-open subsets A and B of Y such that $A\cap B=\emptyset$ and $Y=A\cup B$. Thus, $U=Y-A$ and $V=Y-B$ are nonempty (i,j)-α^m-closed subsets of Y such that $U\cap V=(Y-A)\cap(Y-B)=Y-(A\cup B)=Y-Y=\emptyset$ and $U\cup V=(Y-A)\cup(Y-B)=Y-(A\cap B)=Y-\emptyset=Y$. Since f is a contra (i,j)-α^m-irresolute function, we have $f^{-1}(U)$ and $f^{-1}(V)$ are (i,j)-α^m-open subsets of X and also, $f^{-1}(U)\cap f^{-1}(V)=f^{-1}(U\cap V)=f^{-1}(\emptyset)=\emptyset$ and $f^{-1}(U)\cup f^{-1}(V)=f^{-1}(U\cup V)=f^{-1}(Y)=X$. This contradicts the fact that $(X,\tau_1,\tau_2))$ is an (i,j)-α^m-connected space. Therefore, (Y,σ_1,σ_2) is (i,j)-α^m-connected.

The proof of (2) is similar to (1). □

Theorem 9. *A bitopological space (X,τ_1,τ_2) is (i,j)-α^m-connected, if each contra (i,j)-α^m-continuous function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is a T_0-space, then f is a constant function.*

Proof. Suppose that (X,τ_1,τ_2) is not a (i,j)-α^m-connected space and each contra (i,j)-α^m-continuous function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, where (Y,σ_1,σ_2) is a T_0-space, is a constant function. Since (X,τ_1,τ_2) is not (i,j)-α^m-connected, then there exists a nonempty proper subset A of X which is both (i,j)-α^m-open and (i,j)-α^m-closed. Let $Y=\{a,b,c\}$, $\sigma_1=\{\emptyset,Y,\{a\},\{b\},\{a,b\}\}$ and $\sigma_2=\{Y,\emptyset,\{a\},\{b\},\{a,b\},\{a,c\}\}$ be a topologies on Y and $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a function such that $f(A)=\{a\}$ and $f(X-A)=\{b\}$. Then f is a non-constant contra (i,j)-α^m-continuous function such that (Y,σ_1,σ_2) is a T_0-space, which is a contradiction. Therefore, (X,τ_1,τ_2) is an (i,j)-α^m-connected space. □

4. (i,j)-α^m-compactness

In this section, we introduce and study the concepts of (i,j)-α^m-compact space and (i,j)-α^m-C-compact space. Throughout this section (X,τ_1,τ_2) is a bitopological space where $(i,j)\in\{1,2\}$.

Definition 18. *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-α^m-compact if each (i,j)-α^m-cover of X has a finite subcover.*

Theorem 10. *If a bitopological space (X,τ_1,τ_2) is (i,j)-α^m-compact and F is a proper (i,j)-α^m-closed set of X, then each (i,j)-α^m-open cover of F has a finite subcover.*

Proof. Let $U=\{U_\delta:\delta\in\Delta\}$ be an (i,j)-α^m-open cover of F, then $U\cup(X-F)$ is (i,j)-α^m-cover of X. Since X is (i,j)-α^m-compact, $U\cup(X-F)$ has a finite subcover V for X. Now a finite subcover for F can be obtained by V easily. □

Theorem 11. *A bitopological space (X, τ_1, τ_2) is (i,j)-α^m-compact if X is pairwise compact and if for each (i,j)-α^m-cover U of X there exists a pairwise open collection associated to U that is a cover of X.*

Proof. Let U be (i,j)-α^m-cover of X. By hypothesis, we have a pairwise open cover G associated to U. Si for each (i,j)-α^m-open set $A \in U$, there exist an τ_i-open set $B \in G$ such that $B \subset A \subset \tau_j\text{-}Cl(B)$. Since X is pairwise compact, G has a finite subcover and then U has a finite subcover. Therefore, X is (i,j)-α^m-compact □

Theorem 12. *Let (X, τ_1, τ_2) be (i,j)-α^m-compact. Then, for each pairwise open collection G associated to an (i,j)-α^m-cover of X there exits a finite subcollection $H \subset G$ such that $\{\tau_j\text{-}Cl(A) : A \in H \cap \tau_i, i \in \{1,2\}, i \neq j\}$ covers X.*

Proof. Let U be an (i,j)-α^m-cover of X and G be a pairwise open collection associated to U. Since X is (i,j)-α^m-compact, there exits a finite subcover $V = \{V_i : i \in \{1,2,...,n\}\}$ of U. Now, for each (i,j)-α^m-open set $V_i \in V$, there exits an τ_i-open set $G_i \subset G$ such that $G_i \subset V_i \subset \tau_i\text{-}Cl(G_i)$. So $G_n = \{G_i : i \in \{1,2,...,n\}\}$ is a finite subcollection of G. Since V is a subcover of U, this implies that $\{\tau_j\text{-}Cl(G_i) : G_i \in G_n \cap \tau_i, i \in \{1,2\}\}$ covers X. □

Theorem 13. *If A is an (i,j)-α^m-compact subset of an (i,j)-α^m-T_2 space in (X, τ_1, τ_2) and $x \in X - A$, then there is an (i,j)-α^m-open set B such that $A \subset B$ (or there is a set G such that $x \in G \subset X - B$).*

Proof. Suppose that A is an (i,j)-α^m-compact subset of X and $x \in X - A$. Then, for each $a \in A$, there exits disjoint (i,j)-α^m-open sets U_x and V_a such that $x \in U_x$ and $a \in V_a$. Then, the collection $\{V_a : a \in A\}$ is an (i,j)-α^m-open covering of A. Since A is (i,j)-α^m-compact, there is a finite subcollection $\{V_{a_1}, V_{a_2}, ...V_{a_n}\}$ of (i,j)-α^m-open sets covering A. Now, let $B = \bigcup_{i=1}^{n} V_{a_i}$. Then, clearly B is (i,j)-α^m-open and $A \subset B$. □

Theorem 14. *(i,j)-α^m-compactness is preserved under α^m-continuous, open and onto functions.*

Proof. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces, and let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a (i,j)-α^m-continuous, open and onto functions. Now, suppose that $U^Y = \{U_\delta : \delta \in \Delta\}$ is an (i,j)-α^m-cover of Y, then $U^X = \{f^{-1}(U_\delta : \delta \in \Delta\}$ is an (i,j)-α^m-cover of X. Since X is (i,j)-α^m-compact, there exits a finite subcover $V^X = \{f^{-1}(U_{\delta_i}) : \delta_i \in \Delta, i = 1,2,...,n\}$ of U^X for X. Now, we have that

$$Y = f(X)$$

$$= f(\bigcup_{i=1}^{n}\{f^{-1}(U_{\delta_i}) : i \in \{1,2,...,n\}\})$$

$$= \bigcup_{i=1}^{n}\{f(f^{-1}(U_{\delta_i})) : i \in \{1,2,...,n\}\}$$

Since f is onto, then

$$Y = \bigcup_{i=1}^{n}\{U_{\delta_i} : i \in \{1,2,...n\}\}$$

Therefore, Y is (i,j)-α^m-compact. □

Definition 19. *Let (X, τ_1, τ_2) be a bitopological space. Then, X is said to be (i,j)-α^m-C-compact if given and (i,j)-α^m-closed set A of X and a cover $\{V_\delta : \delta \in \Delta\}$ of A by (i,j)-α^m-open sets of X, then there exits a finite subset $\Delta 0$ of Δ such that $A \subset \bigcup\{\alpha^m BCl(V_\delta : \delta \in \Delta_0)\}$, where $i \neq j$.*

Theorem 15. *Let Y be (i,j)-α^m-extremally disconnected, (i,j)-α^m-C-compact and (i,j)-α^m-T_2. Then, $f : (X, \tau_i, \tau_j) \to (Y, \sigma_i, \sigma_j)$ is (i,j)-α^m-irresolute if and only if it is (i,j)-α^m-set-connected.*

Proof. **NECESSITY**: The proof is easy following the Definition.

SUFFICIENCY: Let f be not (i,j)-α^m-irresolute. Then, there exits an (i,j)-α^m-closed set J of Y such that $f^{1-}(J)$ is not an (i,j)-α^m-closed set of X. Now, let $x \in \alpha^m BCl(f^{-1}(J)) - f^{-1}(J)$. Then X is (i,j)-α^m-connected between $f^{-1}(J)$ and x. Hence, Y is (i,j)-α^m-connected between $f(f^{-1}(J))$ and $f(x)$. In consequence Y is (i,j)-α^m-connected between J and $f(x)$. Since Y is (i,j)-α^m-T_2, for each $y \in J$ there exits an (i,j)-α^m-open set U_y containing y in Y such that $f(x) \notin \alpha^m BCl(U - y)$. Then, the family $\{U_y : y \in J\}$ is a cover of F by (i,j)-α^m-open sets of Y. Now, since Y is (i,j)-α^m-C-compact, there exist a finite number of points $y_1, y_2, ..., y_n$ in J such that $J \subset \bigcup_{i=1}^{n} \alpha^m BCl(U_{y_i}) = U$. Then, U is (i,j)-α^m-coplen set of Y since Y is (i,j)-α^m-extremally disconnected. Besides, $f(x) \notin U$ since $f(x) \in \alpha^m BCl(U_y)$ for any $y \in J$ an this is a contradiction. Hence f is (i,j)-α^m-irresolute. □

Proposition 3. *Let A, B be (i,j)-α^m-C-compact and $A, B \in (X, \tau_i, \tau_j)$, then $A \cup B$ is (i,j)-α^m-C-compact.*

Proof. Since A and B are (i,j)-α^m-C-compact, If we want to prove that $A \cup B$ is (i,j)-α^m-C-compact, we have to prove that for any τ_j-α^m-open which cover $A \cup B$, has a finite sub-cover. Now, let $\{U_i : i \in I\}$ be any cover of τ_j-α^m-open of $A \cup B$. Then, $A \cup B \subseteq \{\bigcup U_i : i \in I\}$, therefore $A \subseteq \bigcup U_i$ and $B \subseteq \bigcup U_i$, this implies that $\{\bigcup U_i : i \in I\}$ is a τ_j-α^m-open cover of $A \cup B$, where $i \neq j$. But, we know that A, B are (i,j)-α^m-C-compact, there exits $i_1, i_2, ..., i_n \in I$ and $t_1, t_2, ...t_n \in I$ such that $\{U_{i_1}, U_{i_2}, ..., U_{i_n}\}$ and $\{U_{t_1}, U_{t_2}, ..., U_{t_n}\}$ is a finite sub-cover of A and B respectively, indeed $\{U_{i_1}, U_{i_2}, ..., U_{i_n}\} \cup \{U_{t_1}, U_{t_2}, ..., U_{t_n}\}$ is a sub-cover of $A \cup B$. In consequence, $A \cup B$ is (i,j)-α^m-C-compact. □

Theorem 16. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i,j)-α^m-continuous function, then the image of an (i,j)-α^m-C-compact is (i,j)-α^m-C-compact.*

Proof. Let X be (i,j)-α^m-C-compact and let f be an (i,j)-α^m-continuous function. Now, let $\{U_i : i \in I\}$ be any cover τ_j-α^m-open of Y, then $\{f^{-1}(U_i) : i \in I\}$ is a cover τ_j-α^m-open of X which is (i,j)-α^m-C-compact. And so, there exits $i_1, i_2, ..., i_n \in I$ such that $\{f^{-1}(U_{i_j}) : j = 1, 2, ...n\}$ is a cover of X and since f is (i,j)-α^m-continuous, then $\{U_{i_j} : j = 1, 2, ...n\}$ is a finite sub-cover of Y. Therefore, Y is (i,j)-α^m-C-compact. □

Theorem 17. *Let A be τ_i-closed of a (i,j)-α^m-C-compact space X, then A is (i,j)-α^m-C-compact.*

Proof. Let A be τ_i-closed of a (i,j)-α^m-C-compact space X and let $\sqcap = \{V_\delta : \delta \in \Delta\}$ a cover τ_j-α^m-open of a subset τ_i-closed B of A. Now, since A is τ_i-closed, $\sqcap$ is a cover τ_j-α^m-open of a subset τ_i-closed B of A. Therefore, $B \subset \bigcup_{i=1}^{n} \tau_i$-$\alpha^m BCl(V_i)$. In consequence A is (i,j)-α^m-C-compact. □

Definition 20. *A topological space (X, τ) is said to be α^m-C-compact, if for each subset closed $A \subset X$ and fore each set α^m-open which is a cover $\mathbb{U} = \{U_\delta | \delta \in \Delta\}$ of A, there exits a finite sub-collection $\{U_{\delta_i} | 1 \leq i \leq n\}$ de $\mathbb{U}$, such that $A \subset \bigcup_{i=1}^{n} Cl_{\alpha^m}(U_{\delta_i})$.*

Theorem 18. *Let (X, τ_1, τ_2) be (i,j)-α^m-C-compact, then (X, τ_1) and (X, τ_2) are α^m-C-compact.*

Proof. Let $\{U_i : i \in I\}$ be any cover α^m-open of X, this implies that $\{U_i : i \in I\}$ is a cover α^m-open of X and since X is an (i,j)-α^m-C-compact, thus there exits a finite sub-cover of X. Indeed, (X, τ_1) is α^m-C-compact.

The proof of (X, τ_2) is similar to (X, τ_1). □

Theorem 19. *If X is (i,j)-α^m-C-compact, then X is pairwise C-compact.*

Proof. Let A be a subset τ_j-closed of X, then $X - A$ is τ_j-open in X.It is well known that every τ_j-open set is α^m-open. Then, we have that $X - A$ is α^m-open set of X. Now, let $\sqcap$ a cover τ_j-open of A, then $\sqcap$ is a cover α^m-open de X. But, since X is (i,j)-α^m-C-compact, then there exits a finite sub-family $V_1, V_2, ..., V_n \in \mathcal{C}$ such that $X = V_1 \cup V_2 \cup ... \cup V_n$. Then, $A \subset V_1 \cup V_2 \cup ... \cup V_n \cup (X - A)$, therefore $A \subset V_1 \cup V_2 \cup ... \cup V_n$, where $V_1, V_2, ..., V_n \in \sqcap$. This implies that X is pairwise compact. □

5. Conclusion

In this paper, we have studied the notions of compactness, connected and separation axioms properties by using (i,j)-α^m-open sets in bitopological spaces. The results obtained in this paper can allow to make some extensions or study new properties such that para-compactness. Also, these results can be proved in tritopological spaces.

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